

MAT2000: ∞ -groupoids

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A recent technological revolution in the field of algebraic topology is given by ∞ -categories. Informally, these can be thought of as a variant of categories where in addition to objects and morphisms between objects, one also has “2-morphisms” between morphisms, “3-morphisms” between 2-morphisms, and so on.

These are supposed to capture any kind of setting where one has a notion of homotopy between morphisms, for example the category of topological spaces (with 2-morphisms given by homotopies). Somewhat curiously, one can also obtain an ∞ -category from a space X itself, where the objects are given by the points of X , morphisms by paths in X , 2-morphisms by homotopies between paths (and so on).

Historically, implementing a notion of ∞ -categories where these examples work out has been challenging, and different models for ∞ -categories have been developed. One guiding principle has been the following:

Homotopy hypothesis (Grothendieck). *Under the above assignment that assigns to every space X an ∞ -category, one should have a 1-to-1 correspondence between (weak) homotopy equivalence classes of spaces, and equivalence classes of ∞ -groupoids (∞ -categories where all morphisms are invertible.)*

By now, the prevailing model for ∞ -categories is given by *quasicategories*, introduced by Joyal [1] and fleshed out to a powerful theory by Lurie in his massive book [2]. These are defined as a type of simplicial sets, characterized by so-called “horn-filling properties”. In this language, the “homotopy hypothesis” is a theorem of Joyal:

Theorem (Joyal, [1]). A quasicategory in which all morphisms are invertible is a Kan complex.

Here Kan complexes are another type of simplicial sets, characterized by horn-filling properties as well.

This can be viewed as a form of the homotopy hypothesis by an earlier result of Quillen which identifies Kan complexes with topological spaces (up to equivalence).

- Goals.**
- Understand the definition of quasicategories, and enough of their structure to talk about invertibility of morphisms.
 - Understand the proof of Joyal’s theorem. Joyal’s proof goes through some clever intermediate notions, it could be interesting to try and find a more

combinatorial proof, or at least try and see explicitly what happens in low degrees (e.g., explicitly find fillers of outer 3-horns).

- Understand the relationship between Kan complexes and topological spaces.

References

- [1] A. Joyal. Quasi-categories and Kan complexes. volume 175, pages 207–222. 2002. Special volume celebrating the 70th birthday of Professor Max Kelly.
- [2] Jacob Lurie. *Higher Topos Theory*, volume 170 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2009.