

Project Proposal:

LATTICE POINTS IN POLYTOPES

Jules Chenal*
Universitetet i Oslo

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In this project, we invite the student to study the properties of the number of lattice points in a lattice polytope. A *lattice point* is a point with integer coordinates of the Euclidean space $\mathbb{R}^n$ and a *lattice polytope* is the convex hull P of a finite number of lattice points. The goal is then to understand the function $k \mapsto L(P; k)$ that counts how many lattice points lie in the image of P by the homothety centred at the origin and of non-negative integral ratio k . In formula, we have:

$$L(P; k) := \text{card}(k \cdot P) \cap \mathbb{Z}^n.$$

Ehrhart showed this quantity to be polynomial, i.e. there exists a unique polynomial – the *Ehrhart polynomial* of P – such that $E(P; k)$ agrees with $L(P; k)$ for all non-negative integers k , cf. [Ehrhart, 1962] or [Beck and Robins, 2015]. In addition of proving this result we will also show the degree of the Ehrhart polynomial to equal the dimension of the polytope, and its leading coefficient to be the integral volume of the polytope. As a first step, we would study Pick’s Theorem cf. [Aigner and Ziegler, 2018]. The latter establishes a relation between the area a of a planar integral polygonal domain D , the number of lattice points i contained in its interior, the number of lattice points b contained on its boundary, and its number of holes h . The relation is the following:

$$a = i + \frac{1}{2}b + h - 1.$$

It can be used to deduce Ehrhart’s Theorem for planar polytopes.

All these results are quite elementary but allow to briefly tackle the counting technique of generating series as well as some concepts of algebraic topology.

Keywords: Combinatorics; Lattice Geometry; Generating Series; Polynomials; Triangulation; Euler Characteristic.

References

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*julesche{at}math.uio.no