

MAT2000 PROJECTS

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1. PERFECT CUBOIDS AND VARIANTS

Is there a *perfect cuboid*? That is, does there exist a rectangular box where all edges, face diagonals, and the space diagonal have *integer* lengths?

This is a famous problem in mathematics, going back to Euler (see R. Guy *Unsolved problems in number theory*).

The project will investigate some variants of this problem.

- Given a prime number p , are there cuboids which are perfect modulo p ? Or modulo p^n ? What about the finite field with p^n elements? How many?

Here there should be room for some counting using e.g., Python.

- Can four squares be found such that the sum of any pair is a perfect square?
- Can four squares be found such that the difference of any pair is a perfect square?
- Is there a point all of whose distances from the corners of the unit square are rational?

2. PYTHAGORAS NUMBERS OF POLYNOMIALS

Let $P(x_1, \dots, x_n)$ be a polynomial of even degree $2d$. What is the minimal integer r such that there exist polynomials $Q_1(x_1, \dots, x_n), \dots, Q_r(x_1, \dots, x_n)$ such that

$$P(x_1, \dots, x_n) = Q_1(x_1, \dots, x_n)^2 + \dots + Q_r(x_1, \dots, x_n)^2.$$

How do we find this integer r ? How do we find such $Q_1, \dots, Q_r$? What if the Q_i are allowed to be rational functions?

The project will survey these questions from the perspective of algebra and algebraic geometry.

3. $\mathbb{Q}$ -DERIVED POLYNOMIALS

Let $q(x) \in \mathbb{Q}[x]$ be a polynomial. We say that $q(x)$ is $\mathbb{Q}$ -*derived* if $q(x)$ and all its derivatives have all roots in $\mathbb{Q}$. A polynomial is said to be of type $p_{m_1, \dots, m_r}$ if it has r distinct roots with multiplicities $m_1, \dots, m_r$.

The classification of $\mathbb{Q}$ -derived polynomials is reduced to the following conjecture:

Conjecture: No polynomial of type $p_{1,1,1,1}$ is $\mathbb{Q}$ -derived.

If this conjecture holds, then every $\mathbb{Q}$ -derived polynomial is equivalent to one of

$$x^n, \quad x^{n-1}(x-1), \quad x(x-1)\left(x - \frac{v(v-2)}{v^2-1}\right), \quad x^2(x-1)\left(x - \frac{9(2w+z-12)(w+2)}{(z-w-18)(8w+z)}\right),$$

where $z, v, w \in \mathbb{Q}$, and $z^2 = w(w-6)(w+18)$.

The project will be devoted to surveying this problem and its variants.

4. INTRODUCTION TO ALGEBRAIC GEOMETRY

Many different topics are possible (toric varieties, Riemann surfaces, plane curves, singularities, ...). Details can be discussed later.