

Hitting times for diffusion processes

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1 Background and outline of the problem

The Itô stochastic differential equation (SDE)

$$dX(t) = a(X(t))dt + b(X(t))dW(t),$$

where $a : \mathbb{R}^d \rightarrow \mathbb{R}^d$ and $b : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$ and $W(t)$ denotes a d -dimensional standard Wiener process, is a diffusion process used to model the evolution of stock prices and the motion of heavy particles immersed in a heat bath of light particles, among other things. Well-posedness results have been established when the coefficients a and b have sufficient regularity, but it is challenging to develop efficient numerical methods for solving many SDE, for instance hitting-time problems.

For a given open, bounded and connected domain $D \subset \mathbb{R}^d$ with a smooth boundary, the above SDE may be associated to the following linear parabolic partial differential equation (PDE) depending on the same coefficients a and b :

$$\begin{aligned} u_t(x, t) + a(x) \nabla u(x, t) + \frac{1}{2} \sum_{i,j,k=1}^d b_{ik} b_{kj} u_{x_i x_j}(x, t) &= 0 & (x, t) \in D \times [0, T) \\ u(x, T) &= g(x) & x \in D \\ u(x, t) &= g(x) & (x, t) \in \partial D \times [0, T]. \end{aligned}$$

The Feynman–Kac formula tells us that the SDE paths can be viewed as stochastic characteristic functions of the PDE in the following sense

$$u(x, t) = \mathbb{E}[g(X(\min(\tau, T))) \mid X(t) = x],$$

where $\tau := \min\{s \geq t \mid X(s) \in \partial D\}$, denotes the hitting time: the first time $s \geq t$ that the SDE $X(s)$ hits the boundary of D . The formula's connection between SDEs and PDEs also presents a way to approximate the solution $u(x, t)$ by generating a large number of realizations of the SDE with initial condition $X(t) = x$, and approximating $\mathbb{E}[g(X(\min(\tau, T))) \mid X(t) = x]$ by a sample average of many realizations of $g(X(\min(\tau, T)))$. This is a Monte Carlo method which can be highly efficient when $d \gg 1$, but, due to the low regularity of SDEs and possibly also large cut-off time $T \gg 1$, it can be very challenging to develop suitable numerical integrators for such methods.

2 Project goals

3 Project goals

The primary objective of this project is to analyze and implement Monte Carlo methods for solving boundary value problems via the Feynman-Kac representation, with a specific focus on the numerical treatment of hitting times.

1. **Theoretical Framework:** derive the Feynman-Kac formula linking the parabolic PDE to the stopped diffusion process, utilizing the connections described in Evans [2] and Øksendal [5].
2. **Numerical Implementation and Boundary Error:** Implement a Monte Carlo solver using the **Euler-Maruyama scheme** for simple domains (e.g., the unit ball in $\mathbb{R}^d$), cf. [1, 4].
3. **Convergence Analysis:** Study the weak convergence rate of the implemented method.
4. **Extensions and Robustness:** Investigate the method's performance in challenging regimes. Specifically, analyze the stability and accuracy for large time-horizons ($T \gg 1$) and explore the impact of reduced boundary regularity (e.g., piecewise smooth boundaries like hypercubes) on the convergence rates.

References

- [1] Higham, Desmond, and Peter Kloeden. *An introduction to the numerical simulation of stochastic differential equations*. Society for Industrial and Applied Mathematics, 2021.
- [2] Evans, Lawrence C. *An introduction to stochastic differential equations*. Vol. 82. American Mathematical Soc., 2012.
- [3] Gobet, Emmanuel. "Weak approximation of killed diffusion using Euler schemes." *Stochastic processes and their applications* 87.2 (2000): 167-197.
- [4] Higham, Desmond J., et al. "Mean exit times and the multilevel Monte Carlo method." *SIAM/ASA Journal on Uncertainty Quantification* 1.1 (2013): 2-18.
- [5] Øksendal, Bernt. *Stochastic differential equations: an introduction with applications*. Springer Science & Business Media, 2013.