

MAT2000: Serre classes

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In algebraic topology, two important invariants of a space X are its *homology groups* $H_n(X)$ and its *homotopy groups* $\pi_n(X)$.

The former are typically easy to compute, while the latter are very complicated, and are not completely understood even for easy spaces like spheres S^m .

Probably the biggest breakthrough in understanding homotopy groups was Serre's thesis. In it, he not only develops the first general-purpose tool to compute homotopy groups (the Serre spectral sequence), he also deduces important qualitative observations, for example that all the groups $\pi_n(S^m)$ are finitely generated, and that most of them are in fact finite. For example, the n -th *stable homotopy group*, $\pi_{n+m}(S^m)$ for large m , is finite for $n > 0$.

The main tool to prove things like this is given by *Serre classes*. A Serre class is a nonempty class $\mathcal{C}$ of abelian groups with the property that for any exact sequence

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0,$$

$B \in \mathcal{C}$ if and only if $A, C \in \mathcal{C}$.

Under some additional assumptions on $\mathcal{C}$ and X , Serre proves that $H_n(X) \in \mathcal{C}$ for all $n \leq N$ if and only if $\pi_n(X) \in \mathcal{C}$ for all $n \leq N$, the “Hurewicz mod $\mathcal{C}$ theorem”. The above results about homotopy groups are deduced from analogous results on homology groups by applying this to different Serre classes, for example the Serre class consisting of all finitely generated abelian groups.

- Goals.**
1. Learn about Serre classes and their basic properties and examples.
 2. Learn about the Hurewicz mod $\mathcal{C}$ theorem and understand how the above results are applications thereof. This involves two additional (algebraic) conditions on Serre classes.
 3. Serre does not settle whether these conditions might be vacuous. This is settled in [1], by classifying all Serre classes of abelian groups, and observing that the two conditions are equivalent, but not always satisfied (he gives an example). The full classification seems quite technical, but an interested student could try to understand it and reproduce some of these results.
 4. Serre proves the Hurewicz mod $\mathcal{C}$ theorem by a qualitative analysis of the Serre spectral sequence. While constructing the Serre spectral sequence is quite technical, its structure and main properties can be black-boxed, and

an interested student could use this opportunity to learn about spectral sequences and understand the proof of the Hurewicz mod $\mathcal{C}$ theorem, and how the additional conditions enter exactly. (For example, in [2] or [3])

5. A more category-theoretic perspective on Serre classes is that one may use them to form a category $\text{Ab}/\mathcal{C}$ of “abelian groups mod $\mathcal{C}$ ”, which is an *abelian category* obtained by “localizing” the category of abelian groups. The above story can then be expressed by working with homology and homotopy groups as objects in $\text{Ab}/\mathcal{C}$ instead of Ab . An interested student could use this opportunity to learn some category theory, understand this localisation $\text{Ab}/\mathcal{C}$, and express the results in those terms.

Of course, it is not reasonable to do all of the above: While 1. and 2. are the core of the project, 3., 4. and 5. are different directions an interested student could take this project in.

References

- [1] S. Balcerzyk. On classes of abelian groups. *Fund. Math.*, 51:149–178, 1962/63.
- [2] Allen Hatcher. Spectral sequences. <https://pi.math.cornell.edu/~hatcher/AT/ATch5.pdf>, 2004.
- [3] Edwin H. Spanier. *Algebraic topology*. Springer-Verlag, New York-Berlin, 1981. Corrected reprint.