UNIVERSITETET I OSLO

Det matematisk-naturvitenskapelige fakultet

Examination in:	MAT 2200 — Groups, rings and fields.
Day of examination:	Thursday, June 8, 2006.
Examination hours:	09.00 – 12.00.

This examination set consists of 2 pages.

Appendices:	None.
Permitted aids:	None.

Make sure that your copy of the examination set is complete before you start solving the problems.

Problem 1.

a) Explain why the automorphism group of the additive group $\mathbb{Z}_n$ is isomorphic to the multiplicative group $\mathbb{Z}_n^*$ of units of the ring $\mathbb{Z}_n$.

b) Show that if $p \neq 2$ is prime then $\mathbb{Z}_p^*$ has only one element of order 2. Conclude that the map $a \mapsto -a$ is the only automorphism of order 2 of the additive group $\mathbb{Z}_p$.

c) Let p be a prime, $p \neq 2$, and K a group of order $2p$. What can you say about the number of Sylow 2-subgroups and p-subgroups of K using Sylow's theorems?

Problem 2.

Consider the polynomial $f(x) = x^6 - 2 \in \mathbb{Q}[x]$. Let $L \subset \mathbb{C}$ be the splitting field of f.

a) Explain why f is irreducible.

b) Prove that $[L \colon \mathbb{Q}] = 12$.

(Continued on page 2.)

Problem 3.

Assume a group G acts by automorphisms on a group H, that is, we are given a homomorphism $G \to \mathrm{Aut}(H)$, $g \mapsto \alpha_g$.

a) Explain why $\alpha_g(h^{-1}) = (\alpha_g(h))^{-1}$ and $\alpha_{g^{-1}}(h) = (\alpha_g)^{-1}(h)$ for $g \in G$ and $h \in H$.

Define a binary operation on the set $H \times G$ by

$$(h_1, g_1)(h_2, g_2) = (h_1\alpha_{g_1}(h_2), g_1g_2).$$

b) Show that $H \times G$ with the above operation is a group, which is called a semidirect product of H and G. (Hint: $(h,g)^{-1} = (\alpha_{g^{-1}}(h^{-1}), g^{-1})$.) We shall denote this group by $H \rtimes_\alpha G$.

c) Show that the maps $H \ni h \mapsto (h, e) \in H \rtimes_\alpha G$ and $G \ni g \mapsto (e, g) \in H \rtimes_\alpha G$ are injective homomorphisms of groups. Therefore we can consider H and G as subgroups of $H \rtimes_\alpha G$. Show that H is normal in $H \rtimes_\alpha G$, and $(H \rtimes_\alpha G)/H$ is isomorphic to G.

d) Assume G and H are subgroups of a group K such that

   (i) K is generated by G and H;

   (ii) $G \cap H = \{e\}$;

   (iii) H is normal in K.

   For $g \in G$ define an automorphism α_g of H by $\alpha_g(h) = ghg^{-1}$. Show then that the map $H \rtimes_\alpha G \to K$, $(h, g) \mapsto hg$, is an isomorphism of groups.

Problem 4.

Consider the group $G = \mathbb{Z}_2$. Notice that to have an action of $\mathbb{Z}_2$ on a group H by automorphisms is the same as to be given an automorphism β of H such that $\beta^2 = \mathrm{id}$. We then say that β is an involutive automorphism of H. If $H = \mathbb{Z}_n$ then the map $a \mapsto -a$ on $\mathbb{Z}_n$ is such an automorphism. The corresponding semidirect product of $\mathbb{Z}_n$ and $\mathbb{Z}_2$ is called the dihedral group of order $2n$ and denoted by D_n. Therefore, if we use multiplicative notation, D_n is a group generated by two elements x and y such that $x^n = e$, $y^2 = e$ and $yxy^{-1} = x^{-1}$.

a) Let K be as in Problem 1(c). Use 1(b), 1(c) and 3(d) to show that K is either cyclic or isomorphic to the dihedral group D_p.

b) Let L be as in Problem 2. Show that the Galois group $G(L/\mathbb{Q})$ is isomorphic to the dihedral group D_6.

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