UNIVERSITETET I OSLO

Det matematisk-naturvitenskapelige fakultet

Examination in:	MAT2200 — Groups, Rings and Fields.
Day of examination:	Thursday 7. Juni 2007.
Examination hours:	09.00 – 12.00.
This examination set consists of 2 pages.	
Appendices:	None.
Permitted aids:	None.

Make sure that your copy of the examination set is complete before you start solving the problems.

Exercise 1. Let $M_2(\mathbb{Z}_3)$ be the ring of 2×2-matrices over the field $\mathbb{Z}_3$ and let G be the subring of $M_2(\mathbb{Z}_3)$ given by

$$G = \{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mid a, b \in \mathbb{Z}_3 \},$$

where G has the operations of matrix addition and multiplication from $M_2(\mathbb{Z}_3)$.

a) Prove that G is a field.

b) Show that the polynomial $x^2 + 1$ is irreducible over $\mathbb{Z}_3$ and explain why $\mathbb{Z}_3[x]/\langle x^2 + 1 \rangle$ becomes a field.

c) Let α be a zero of x^2+1 in an extension field of $\mathbb{Z}_3$. Remember that $\mathbb{Z}_3(\alpha)$ is by definition isomorphic to $\mathbb{Z}_3[x]/\langle x^2 + 1 \rangle$. Show that the map

$$\phi(\begin{pmatrix} a & b \\ -b & a \end{pmatrix}) = a + \alpha b$$

is an isomorphism from G onto $\mathbb{Z}_3(\alpha)$.

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Exercise 2.

a) Show that the polynomial $f(x) = x^5 - 2$ is irreducible over $\mathbb{Q}$.

b) Let $K = \mathbb{Q}(\sqrt[5]{2}, \zeta)$ where $\sqrt[5]{2}$ is the real 5-root of 2 and ζ is a primitive 5-root of unity ($= \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$). Show that K is the splitting field of $f(x)$ over $\mathbb{Q}$ and that $[K : \mathbb{Q}] = 20$.

c) Explain why there is only one intermediate field $\mathbb{Q} \leq K' \leq K$ with K' normal over $\mathbb{Q}$ and $[K' : \mathbb{Q}] = 4$.

d) Find the number of intermediate fields $\mathbb{Q} \leq E \leq K$ such that $G(K/E)$ corresponds to Sylow 2-subgroups of $G(K/\mathbb{Q})$.

Exercise 3. Let $\alpha = \sqrt{5 - 2\sqrt{5}}$.

a) Find the irreducible polynomial $f(x)$ of α over $\mathbb{Q}$.

b) Let K be the splitting field of $f(x)$ over $\mathbb{Q}$. Explain why $[K : \mathbb{Q}] = 4$. Show that $[K : \mathbb{Q}(\sqrt{5})] = 2$.

c) Show that the Galois group $G(K/\mathbb{Q})$ is isomorphic to the cyclic group $(\mathbb{Z}_4, +)$. (Hint: note that one of the elements β in K that is conjugate with α over $\mathbb{Q}$ satisfies $\alpha\beta = \sqrt{5}$ and consider extensions to $G(K/\mathbb{Q})$ of the non-trivial element in $G(\mathbb{Q}(\sqrt{5})/\mathbb{Q})$).

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