UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in MAT2200 — Groups, rings and fields

Day of examination: 5. June 2009

Examination hours: 09.00 – 12.00

This problem set consists of 2 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Note: You must give full reasons for all your answers!

Problem 1

1a

Find all abelian groups of order 63.
One of these groups is cyclic. Determine all its subgroups.

1b

Let G be a nonabelian group of order 21. What are the possible orders of elements of G?

Find the number of Sylow p–subgroups of G, for each prime p.

Determine how many elements G has of each order.

Does G have any other proper, nontrivial subgroups?

1c

Let s be an element of order 3 and t an element of order 7 in G. Show that we must have $sts^{-1} = t^{\ell}$, for some natural number ℓ.
What are the possible values of ℓ?

Problem 2

2a

Let $f(x) = x^4 - 3 \in \mathbb{Q}[x]$. Explain why $f(x)$ is irreducible over $\mathbb{Q}$.
Show that the splitting field K of $f(x)$ over $\mathbb{Q}$ is $\mathbb{Q}(\sqrt[4]{3}, i)$, where $i^2 = -1$.

2b

Find $[K : \mathbb{Q}]$. Show that the Galois group $G(K/\mathbb{Q})$ has a normal subgroup which is cyclic of order 4, and describe a generator of this subgroup.

(Continued on page 2.)

2c

Observe that the roots of $f(x)$ are vertices of a square in the complex plane. Use this to show that $G(K/\mathbb{Q})$ is isomorphic to the dihedral group D_4.

Problem 3

3a

Let R be a commutative ring with unit element 1. If J_1 and J_2 are two ideals, we define

$$J_1 + J_2 = \{a_1 + a_2 \in R \,|\, a_1 \in J_1 \text{ and } a_2 \in J_2\}$$

Show that $J_1 + J_2$ is an ideal of R.

Define $\Phi : R \to R/J_1 \times R/J_2$ by $\Phi(a) = (a + J_1, a + J_2)$. Show that Φ is a ring homomorphism and that the kernel of Φ is $J_1 \cap J_2$.

3b

We say that J_1 and J_2 are *relatively prime* if $J_1 + J_2 = R$. Show that J_1 and J_2 are relatively prime if and only if we can find $a_1 \in J_1$ and $a_2 \in J_2$ such that $a_1 + a_2 = 1$.

Show that if n and m are two integers, the ideals $\langle n \rangle$ and $\langle m \rangle$ are relatively prime if and only if n and m are relatively prime in the usual sense.

3c

Assume now that the ideals J_1 and J_2 of R are relatively prime. Let r_1 and r_2 be elements of R. Show that we there is an $r \in R$ such that $r - r_1 \in J_1$ and $r - r_2 \in J_2$.

Show that the homomorphism Φ of 3a gives rise to an isomorphism of rings

$$R/(J_1 \cap J_2) \simeq R/J_1 \times R/J_2 \,.$$

END