# UNIVERSITY OF OSLO

## Faculty of Mathematics and Natural Sciences

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| Examination in | MAT2200 — Groups, rings, and fields. |
| Day of examination: | Wednesday, June 2, 2010. |
| Examination hours: | 14.30 – 17.30. |
| This problem set consists of 2 pages. | |
| Appendices: | None. |
| Permitted aids: | None. |

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

**Note: You must write proofs for all your answers!**

## Problem 1

Let $M_3(\mathbb{Z}_5)$ be the ring of $3 \times 3$ matrices over the field $\mathbb{Z}_5$. Let $R$ denote the following subset of $M_3(\mathbb{Z}_5)$:

$$R = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} \mid a, b, c \in \mathbb{Z}_5 \right\}.$$

### 1a

Show that $R$ with the operations of addition and multiplication of matrices from $M_3(\mathbb{Z}_5)$ is a commutative subring of $M_3(\mathbb{Z}_5)$ with unity.

### 1b

Show that the map $\phi : R \to \mathbb{Z}_5$ given by

$$\phi \begin{pmatrix} a & b & c \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} = a$$

is a homomorphism of rings.

### 1c

Show that the subset

$$I = \left\{ \begin{pmatrix} 0 & b & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid b, c \in \mathbb{Z}_5 \right\}.$$

*(Continued on page 2.)*

is an ideal of $R$, and that $R/I$ is a field. Find the number of elements in $R/I$.

# Problem 2

Let $G$ be a group of order $|G| = 595 = 5 \cdot 7 \cdot 17$.

## 2a

Formulate Sylow's third theorem. Find the possible numbers of Sylow $p$-subgroups of $G$ for each of the primes $5, 7, 17$.

## 2b

Show that $G$ has a normal Sylow 5-subgroup and a normal Sylow $p$-subgroup for at least one of the primes $p \in \{7, 17\}$.

## 2c

Let $K$ be the normal Sylow 5-subgroup of $G$. Show that $G/K$ is abelian.

# Problem 3

Let $f(x) = x^3 - 3$ and $g(x) = x^4 - 2x^2 - 3$ in $\mathbb{Q}[x]$.

## 3a

Show that $f(x)$ is irreducible and $g(x)$ is reducible over $\mathbb{Q}$.

## 3b

Find the splitting field $E$ of $g(x)$. Compute $[E : \mathbb{Q}]$ and find, up to isomorphism, the Galois group $G(E/\mathbb{Q})$.

## 3c

Find the splitting field $K$ of the family $\{f(x), g(x)\}$. Show that $[K : \mathbb{Q}] = 12$.

END.