UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in MAT2200 — Groups, rings, and fields.

Day of examination: Wednesday, June 6, 2012.

Examination hours: 14.30 – 18.30.

This problem set consists of 2 pages.

Appendices: None.

Permitted aids: None.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Note: Justify all answers!

Problem 1

Consider the field $\mathbb{Z}_p$ for p a prime and let G denote the following subset of the 2×2 matrices $M_2(\mathbb{Z}_p)$:

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid b \in \mathbb{Z}_p, a, c \in \mathbb{Z}_p \setminus \{0\} \right\}.$$

1a

Show that G with the operation of multiplication of matrices from $M_2(\mathbb{Z}_p)$ is a group. Find the order of G.

1b

Let $p = 3$. Prove that G has at least one normal subgroup which is different from G and the trivial subgroup. (For example by finding one such subgroup or by using Sylow's theorems.)

Let G' be a group of order 12. Let H denote a Sylow 2-subgroup of G' and K a Sylow 3-subgroup of G'.

1c

Assume that both H and K are normal subgroups of G'. Show that $hk = kh$ for all $h \in H$ and $k \in K$. Conclude that the map $\phi : H \times K \to G'$, $\phi(h, k) = hk$ for $(h, k) \in H \times K$ is an isomorphism of groups, and that G' is abelian.

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1d

Assume that the Sylow 3-subgroup K is normal and the Sylow 2-subgroup H is not normal in G'. Show that there exist elements $h \in H$ and $k \in K$ such that $hkh^{-1} = k^2$.

Problem 2

Let R and R' be rings, and let $\phi : R \to R'$ be a ring homomorphism.

2a

Let N be an ideal of R and N' an ideal of R'. Suppose that $\phi[N] \subseteq N'$. Show that $\psi : R/N \to R'/N'$ given by $\psi(a+N) = \phi(a)+N'$ is a well-defined ring homomorphism. If ϕ is surjective and $\ker\phi \subseteq N$, show that ψ is a ring isomorphism from R/N onto $R'/\phi[N]$.

2b

Assume that R and R' are commutative and have unity 1 and $1'$, respectively, and assume that ϕ is surjective with $\phi(1) = 1'$. Prove that if M is a maximal ideal in R such that $\ker\phi \subseteq M$, then $\phi[M]$ is a maximal ideal in R'.

2c

Let p be a prime. Find a maximal ideal in the ring $\mathbb{Z}_{p^n}$ for every $n \geq 1$.

Problem 3

3a

Let $\alpha = \sqrt{2}+i$ in $\mathbb{C}$. Show that α is algebraic over $\mathbb{Q}$ by finding the irreducible polynomial $f(x)$ for α over $\mathbb{Q}$. Make sure to prove that $f(x)$ is irreducible over $\mathbb{Q}$.

3b

Find the splitting field K of $f(x)$ over $\mathbb{Q}$. Compute $[K : \mathbb{Q}]$ and find, up to isomorphism, the Galois group $G(K/\mathbb{Q})$.

3c

Set up the Galois correspondence between subgroups of $G(K/\mathbb{Q})$ and intermediate subfields $\mathbb{Q} \leq E \leq K$.

END.