UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in: MAT2200 — Groups, Rings and Fields

Day of examination: Wednesday June 5. 2013

Examination hours: 14.30 – 18.30

This problem set consists of 2 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

All subproblems have the same weight.

Note: You must give full reasons for all your answers!

Problem 1

Let G be a finite group and let $\phi\colon G \to G$ be the mapping $x \mapsto x^n$ where n is a natural number.

1a

Show that if G is abelian, then the mapping ϕ is a group homomorphism. Show that if $n = 2$, and ϕ is a group homomorphism, then G is abelian.

1b

Assume that G is cyclic and of even order, and let $n = 2$. Show that $\operatorname{Im}\phi$ is of index two. Show by giving an example, that the condition G be cyclic, is necessary.

1c

Assume that G is cyclic and of even order, and let $n = 6$. What is the index of $\operatorname{Im}\phi$?

Problem 2

2a

Let A be an integer and let $f(X)$ be the polynomial $f(X) = X^4 + AX + 1$. Show that $f(X)$ is irreducible in $\mathbb{Z}[X]$ if and only if $A \neq \pm 2$. Find a factorization of $f(X)$ into irreducible polynomials when $A = \pm 2$.

(Continued on page 2.)

2b

Suppose that F a field where the equation $x^2+1=0$ does not have a solution, but where there is an element $a \in F$ such that $a^2 = -2$. Let $g(X) = X^4+1$. Show that $g(X)$ is reducible in the polynomial ring $F[X]$, and write $g(X)$ as a product of polynomials irreducible in $F[X]$.

2c

Show that X^4+1 is not irreducible over $\mathbb{Z}_{11}$.

Problem 3

Assume that p is a prime such that $p+2$ is a prime as well. Let G be a group of order $p^2(p+2)$. Show that all the Sylow subgroups of G are abelian and normal.

Problem 4

Let S_6 denote the symmetric group on 6 letters.

4a

Show that the cycles $\sigma = (1,2,3)$ and $\tau = (4,5,6)$ generate an abelian subgroup of order 9 of S_6 which is isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_3$.

4b

Show that all the Sylow 3-subgroups of S_6 are isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_3$.

4c

How many Sylow 3-subgroups are there in S_6? HINT: In how many ways can one write $\{1,2,3,4,5,6\}$ as a disjoint union of two subsets each having three elements?

END