UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in: MAT2200 — Groups, Rings and Fields

Day of examination: Friday June 13, 2014

Examination hours: 14.30 – 18.30

This problem set consists of 3 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

All subproblems have the same weight.

Note: You must give full reasons for all your answers!

Problem 1

Let n be a natural number and let S_n denote the symmetric group on n letters. If $\rho \in S_n$, one has the formula

$$\rho(1, 2, \ldots, n)\rho^{-1} = (\rho(1), \rho(2), \ldots, \rho(n)).$$

You may use it without proof.

1a

Show that every n-cycle $(a_1, \ldots, a_n)$ is conjugate to $(1, 2, \ldots, n)$. Explain why there are $(n-1)!$ different n-cycles in S_n.

1b

Assume that $\sigma \in S_n$ is an n-cycle. Let $C(\sigma) = \{\, \rho \in S_n \mid \rho\sigma = \sigma\rho \,\}$. Show that $C(\sigma) = \langle\, \sigma \,\rangle$. Hint: Consider the action of S_n on it self by conjugation and use the relation between the order of an isotropy group and the number of elements in an orbit.

Problem 2

Let σ be the 5-cycle $\sigma = (1, 2, 3, 4, 5)$ and let τ denote the permutation $\tau = (1, 5)(2, 4)$.

2a

Show that $\tau^2 = \mathrm{id}$, that $\tau(3) = 3$ and that $\tau\sigma\tau = \sigma^{-1}$.

(Continued on page 2.)

2b

Let N be the subset of S_5 given by

$$N = \{ \rho \in S_5 \mid \rho\sigma\rho^{-1} = \sigma^{\epsilon(\rho)} \text{ where } \epsilon(\rho) \in \{\pm 1\} \}$$

Show that N is a subgroup of S_5 and that the group $\langle \sigma \rangle$ generated by σ is a normal subgroup of N.

2c

Show that $\epsilon\colon N \to \{\pm 1\}$ is a surjective group homomorphism. Determine the kernel of ϵ and show that N is of order 10. HINT: Problem 1b) might be useful.

Problem 3

3a

Assume that G is a group having two normal, abelian subgroups A and B. Assume that the orders $|A|$ and $|B|$ are relatively prime. Show that $A \cap B = \{e\}$ and that $AB = \{ xy \mid x \in A, y \in B \}$ is an abelian subgroup of order $|A||B|$.

3b

Show that if A and B in the previous subproblem are cyclic, then AB is cyclic.

3c

Let G be a group of order 1001. Show that G is cyclic. HINT: $1001 = 7 \cdot 11 \cdot 13$.

Problem 4

Let K be a field whose characteristic is different from 2. Let L be an extension of K. Assume there is an element $i \in L$ such that $i^2 = -1$ and an element $a \in L$ such that $a^2 = i$.

4a

Show that $a^4 = -1$, that $((1+i)a)^2 = -2$ and that $((1-i)a)^2 = 2$.

4b

Show that the following equality holds in the polynomial ring $L[x]$:

$$x^4 + 1 = (x-a)(x+a)(x-ia)(x+ia)$$

4c

Show that $x^4 + 1$ is irreducible over K if and only if non of -1, 2 or -2 are squares in K.

(Continued on page 3.)

4d

Show that x^4+1 is reducible over each of the fields $\mathbb{Z}_5$ and $\mathbb{Z}_7$. In both cases determine the factorization of x^4+1 in irreducible factors.

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