UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in: MAT2200 — Groups, Rings and Fields

Day of examination: 14. June 2016

Examination hours: 14.30 – 18.30

This problem set consists of 3 pages.

Appendices: none

Permitted aids: none

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

All problems have equal weight.

Problem 1

1a

Find all abelian groups of order 12 up to isomorphism.

1b

Show that the matrices

$$M = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \quad \middle| \quad a, c \in \mathbb{Z}_3 \setminus \{0\},\ b \in \mathbb{Z}_3 \right\}$$

form a group under matrix multiplication.
What is the order of M? Find and list all elements in M of order 3. Is M abelian? Give reasons for your answer.

1c

A permutation is even (resp. odd) if it is a product of an even (resp. odd) number of transpositions.
Show that a 3-cycle is even, and that a 4-cycle is odd.

(Continued on page 2.)

1d

The set of even permutations on 4 elements is a group of order 12. Is it isomorphic to M? Give reasons for your answer.

Problem 2

Let K be the splitting field of $f = (x^3 - 8)(x^2 - 2) \in \mathbb{Q}[x]$.

2a

Find the degree $[K : \mathbb{Q}]$ and the group $G(K/\mathbb{Q})$.

2b

Find an element $a \in K$ such that $K = \mathbb{Q}(a)$.

Problem 3

3a

What is an integral domain? Show that $\mathbb{Z}[x]/(x^2 + 1)$ is a integral domain.
Is $\mathbb{Z}[x]/(x^2 - 1)$ an integral domain? Give reasons for your answer.

Problem 4

An element $r \neq 0$ in a ring is nilpotent if $r^n = 0$ for some $n > 1$.

4a

Show that the set of nilpotent elements in a commutative ring together with 0 form an ideal.

(Continued on page 3.)

4b

The set of matrices

$$M' = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad | \quad a, b, c, d \in \mathbb{Z}_3 \right\}$$

with matrix addition and matrix multiplication is a ring.
Show that a nilpotent element in M' has determinant $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = 0$.
Find and list all nilpotent elements in M'.

END