UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: MAT2200 — Grupper, ringer og kropper

Day of examination: Monday, June 12, 2017

Examination hours: 14.30 – 18.30

This problem set consists of 3 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1

Let F be a field and consider the set of matrices

$$U(F) = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix} : a, b, c \in F \right\}.$$

a

Show that $U(F)$ is a group under matrix multiplication. Is it abelian?

b

The group $U(F)$ has a subgroup

$$H = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & a & 1 \end{bmatrix} : a, b \in F \right\}.$$

Show that H is abelian and normal. If $F = \mathbb{Z}_2$ which group is H?

c

Set $U = U(\mathbb{Z}_2)$. Let $X \subset \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ be the set $X = \{(1, y, z) : y, z \in \mathbb{Z}_2\}$. Show that U acts on X and that the action induces an injective group homomorphism $U \to S_4$ where S_4 is the permutation group of sets with 4 elements. Which subgroup of S_4 is it?

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Problem 2

a

Let H and K be normal subgroups of a group G such that $H \cap K = \{e\}$. Prove that there is an injective group homomorphism $H \times K \to G$.

b

Use the Sylow theorems and the classification of finite abelian groups to find (up to isomorphism) all groups of order 99. (You may use without proof that if p is a prime number and $|G| = p^2$ then G is abelian.)

Problem 3

Let ω be the complex number $\omega = e^{\frac{2\pi i}{12}} = \frac{\sqrt{3}}{2} + \frac{1}{2}i$. Let $f(x) = x^6 + 1 \in \mathbb{Q}[x]$. Note that if α is a zero for $f(x)$ then so is α^{2k+1} for any integer k and that $-\alpha = \alpha^7$.

a

Show that $f(x) = h(x)g(x)$ in $\mathbb{Q}[x]$ where $h(x)$ has degree 2 and $g(x)$ has degree 4. (Hint: The complex number i is a zero for $f(x)$.)

b

Show that $\mathbb{Q}(\omega)$ is a splitting field for $f(x)$ and that $\mathbb{Q}(\omega) = \mathbb{Q}(\sqrt{3}, i)$. Compute $[\mathbb{Q}(\omega) : \mathbb{Q}]$ and the Galois group $G(\mathbb{Q}(\omega)/\mathbb{Q})$. Explain why we may conclude that $g(x)$ is irreducible.

Problem 4

a

Let F be a field and assume that $f(x) \in F[x]$ is an irreducible polynomial of degree n. Let K be a splitting field for $f(x)$. Explain why $n \leq [K : F] \leq n!$. If n is odd and there exists $\delta \in K$ with $\delta^2 \in F$ but $\delta \notin F$, show that $2n \leq [K : F]$.

b

Assume now that $f(x)$ is an irreducible degree 3 polynomial in $\mathbb{Q}[x]$ and let K be a splitting field for $f(x)$. Let α_1, α_2 and α_3 be the zeroes of $f(x)$ in K. Let S be the permutation group of $\{\alpha_1, \alpha_2, \alpha_3\}$ and let $G = G(K/\mathbb{Q})$ be the Galois group. We may think of G as a subgroup of S.

Define

$$\delta = (\alpha_2 - \alpha_1)(\alpha_3 - \alpha_1)(\alpha_3 - \alpha_2) \in K$$

and set $D = \delta^2$.

Show that if $\sigma \in G$ then $\sigma(D) = D$. Explain why this implies that $D \in \mathbb{Q}$. Prove that if D is not a square in $\mathbb{Q}$ then $[K : \mathbb{Q}] = 6$ and $G \simeq S$.

(Continued on page 3.)

c

Let τ be a transposition in S. Prove that if D is a square in $\mathbb{Q}$ then $\tau \notin G$. (Hint: What does τ do to δ?) Use this to show that if D is a square in $\mathbb{Q}$ then $[K : \mathbb{Q}] = 3$ and $G \simeq \mathbb{Z}_3$.

END