UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in:	MAT2200 — Groups, Rings and Fields
Day of examination:	Tuesday, June 5, 2018
Examination hours:	14:30 – 18:30
This problem set consists of 3 pages.	
Appendices:	None
Permitted aids:	None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

All subproblems (1a, 1b,...) carry the same weight.

Problem 1

The least common multiple of the orders of the elements of a group G is called the *exponent* of G.

1a

Find all abelian groups, up to isomorphism, of order 27 and exponent 3. Can you do this without using the classification of finite abelian groups? Find all abelian groups of order 81. Justify your answer.

1b

Show that

$$H = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{Z}_3 \right\} \subset \mathrm{GL}_3(\mathbb{Z}_3)$$

is a nonabelian group of order 27 and exponent 3.

Problem 2

Let G be a group of order 2018.

2a

Show that G contains a normal cyclic subgroup H of order 1009. (You may use without checking that 1009 is a prime number.)

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2b

Show that G is either cyclic or isomorphic to the dihedral group of order 2018. (Recall that the latter group is generated by two elements a and b such that $a^{1009} = e$, $b^2 = e$ and $bab^{-1} = a^{-1}$.) Hint: if $b \in G$ is an element of order 2, then the formula $\alpha(h) = bhb^{-1}$ defines an automorphism α of $H \cong \mathbb{Z}_{1009}$ satisfying $\alpha^2 = \mathrm{id}$.

Problem 3

Let K be a field of characteristic different from 2, $\bar{K}$ be an algebraic closure of K, and $\alpha \in \bar{K} \setminus K$. Let f and g be the irreducible polynomials of α and α^2 over K.

3a

Assume $f(x) \in K[x^2]$, that is, $f(x) = h(x^2)$ for some polynomial $h \in K[x]$. Show that h is irreducible and conclude that $h = g$ and $[K(\alpha) : K(\alpha^2)] = 2$.

3b

Assume now that $f(x) \notin K[x^2]$, so that f contains at least one nonzero term with an odd power of x. Show that $g(x^2)$ is divisible by $f(x)$ and $f(-x)$ and conclude that $K(\alpha) = K(\alpha^2)$.

3c

Take $K = \mathbb{Q}$ and give examples of α, with $\alpha^2 \notin \mathbb{Q}$, illustrating 3a and 3b. In both cases write down the corresponding polynomials $f(x)$ and $g(x)$ and explain why f is irreducible.

Problem 4

Let p be a prime number and $K \subset \mathbb{C}$ be the splitting field of the polynomial $f(x) = x^p - 2 \in \mathbb{Q}[x]$.

4a

Prove that for every natural number $n \geq 2$ we have $[\mathbb{Q}(2^{1/n}) : \mathbb{Q}] = n$.

4b

Consider the pth primitive root of unity $\zeta_p = e^{2\pi i/p}$. Show that $K = \mathbb{Q}(2^{1/p}, \zeta_p)$ and conclude that $[K : \mathbb{Q}] = p(p-1)$. (You may use without explanation that $[\mathbb{Q}(\zeta_p) : \mathbb{Q}] = p - 1$.)

4c

Describe all elements of the Galois group $G(K/\mathbb{Q})$ by their actions on $2^{1/p}$ and ζ_p. Show next that $G(K/\mathbb{Q})$ is isomorphic to the group

$$\left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \in \mathbb{Z}_p^*, b \in \mathbb{Z}_p \right\} \subset \mathrm{GL}_2(\mathbb{Z}_p),$$

(Continued on page 3.)

called the $ax + b$ *group* over the field $\mathbb{Z}_p$.

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