UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: MAT2200 — Groups, rings and fields

Day of examination: Friday 14. June 2019

Examination hours: 9:00 – 13:00

This problem set consists of 2 pages.

Appendices: none

Permitted aids: none

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1 Finite abelian groups

a

Is $\mathbb{Z}_{\geq 0}$ equipped with the operation of addition a group? Justify your answer.

b

What are the abelian groups of order 18 up to isomorphism?

c

What are the cosets of the subgroup $\langle(0,3)\rangle$ in the group $\mathbb{Z}_2 \times \mathbb{Z}_9$? Describe the quotient

$$\frac{\mathbb{Z}_2 \times \mathbb{Z}_9}{\langle(0,3)\rangle}$$

as a product of finite cyclic groups.

Problem 2 Symmetric group

a

Let $\sigma = (2,6,1,4,3)$ and $\tau = (5,2,1,3)$ be permutations in S_6. Express the compositions $\sigma\tau$ in disjoint cycle notation. What is the order of $\sigma\tau$?

b

Show that if $\sigma \in S_n$ is a cycle of odd length $2k+1$ then σ^2 is also a cycle of length $2k+1$.

(Continued on page 2.)

c

Let G be a subgroup of the symmetric group S_n such that there exist permutations $\sigma_2, \ldots, \sigma_n \in G$ satisfying $\sigma_i(1) = i$ for $2 \leq i \leq n$. Show that G acts **transitively** on the set $\{1, \ldots, n\}$. In other words, show that for each pair $1 \leq i, j \leq n$ there is a permutation $\sigma \in G$ such that $\sigma(i) = j$.

Problem 3 Rings and quotients

a

Show that if n is not a prime number, then $n\mathbb{Z}$ is not a prime ideal of $\mathbb{Z}$.

b

Show that the quotient

$$\frac{\mathbb{Z}_2[x]}{\langle x^3 + x + 1 \rangle}$$

is a field. How many elements does it contain?

c

Find a polynomial $p(x) \in \mathbb{Z}_2[x]$ such that $\deg(p(x)) \leq 2$ and

$$x^5 + \langle x^3 + x + 1 \rangle = p(x) + \langle x^3 + x + 1 \rangle$$

in $\frac{\mathbb{Z}_2[x]}{\langle x^3+x+1 \rangle}$.

Problem 4 Galois theory

a

Let K be the splitting field of the polynomial $f(x) = x^3 - 2 \in \mathbb{Q}[x]$ over $\mathbb{Q}$. Find the degree $[K : \mathbb{Q}]$ and the Galois group $G(K, \mathbb{Q})$.

b

Let K be the splitting field of an irreducible polynomial $f(x) \in \mathbb{Q}[x]$ of degree 3 over $\mathbb{Q}$. Let A_3 denote the alternating group, i.e. the subgroup of even permutations in S_3. Show that $G(K, \mathbb{Q}) = A_3$ if and only if $K = \mathbb{Q}(\alpha)$ where α is a root of $f(x)$.

c

Conclude that all roots of the polynomial $f(x)$ from part b) must be in $\mathbb{R}$. Hint: Use that a polynomial of odd degree with coefficients in $\mathbb{Q}$ must have at least one real root $\alpha \in \mathbb{R}$.

THE END