UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in MAT2200 — Groups, rings and fields.

Day of examination: 29 May 2020 at 9:00 to 8 June 2020 at 9:00

Examination hours: 29 May at 9:00 – 8 June at 9:00.

This problem set consists of 4 pages.

Appendices: All.

Permitted aids: All.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Important: This is a take-home exam. You must provide justification for all your answers. You may deliver the solution to the exam written in Norwegian or English. For a short dictionary: field is "kropp", splitting field is "rotkropp", degree of an extension is "tallgrad", homomorphism is "homomorfi".

There are 15 subproblems distributed over 5 main problems. The 15 subproblems have equal weight. To pass the exam, you need to solve correctly 40% of the total of 15 subproblems.

Problem 1

Let S_4 be the permutation group on 4 letters. Denote by ι the identity permutation, and by (i, j) the transposition corresponding to distinct i, j in $\{1, 2, 3, 4\}$. For σ in S_4, we write $\sigma(i, j)$ for the composition of functions $\sigma \circ (i, j)$ in S_4. For instance if σ is the 4-cycle $(1, 3, 2, 4)$ and $(i, j) = (1, 3)$ then the composition $\sigma(i, j)$ is

$$\sigma(1, 3) = \sigma \circ (1, 3) = (1, 2, 4),$$

a 3-cycle. Note that this is typically different from the transposition $(\sigma(i), \sigma(j))$, which in this case would be

$$(\sigma(1), \sigma(3)) = (3, 2).$$

Let $\mu_1 = (1, 2)(3, 4)$, $\mu_2 = (1, 4)(2, 3)$ and $\mu_3 = (1, 3)(2, 4)$ in S_4.

1a

Prove that $N = \{\iota, \mu_1, \mu_2, \mu_3\}$ is a subgroup of S_4. What known group is N isomorphic to?

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1b

Prove that N is a normal subgroup of S_4. You can for example show that for each $\sigma \in S_4$ and each transposition (i, j) in S_4, we have that

$$\sigma(i, j) = (\sigma(i), \sigma(j))\sigma$$

as mappings in S_4, where $(\sigma(i), \sigma(j))$ is again a transposition. Then use this fact to show that

$$\sigma(i, j)(k, l)\sigma^{-1} = (\sigma(i), \sigma(j))(\sigma(k), \sigma(l))$$

for all $\sigma \in S_4$ and any two disjoint transpositions (i, j) and (k, l) in S_4.

Problem 2

Let $M_3(F)$ be the ring of 3×3-matrices over a field F. For $a, b, c \in F$ we denote

$$M(a, b, c) = \begin{pmatrix} a & b & c \\ 0 & a & 0 \\ 0 & 0 & 0 \end{pmatrix} \in M_3(F).$$

Let R be the subset of $M_3(F)$ given by

$$R = \{M(a, b, c) \mid a, b, c \in F\}.$$

2a

Show that R is a subring of $M_3(F)$. Show moreover that R is not commutative and does not contain unity.

2b

Define maps $\phi : R \to F$ and $\psi : R \to F$ by

$$\phi(M(a, b, c)) = a \text{ and } \psi(M(a, b, c)) = c,$$

for $M(a, b, c) \in R$. Prove that ϕ is a ring homomorphism. Is ψ a ring homomorphism?

2c

Show that the subset $I = \{M(a, b, c) \in R \mid a = 0\}$ is an ideal of R. Is I a maximal ideal of R?

Problem 3

Let G be a group of order $|G| = 1225$.

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3a

Show that G has a unique Sylow p-subgroup for each prime p that divides $|G|$. Show further that G has two normal subgroups M and N such that the order of M is co-prime with the order of N. Explain why M and N are abelian (you can here use, without proof, a known result from the course).

3b

Show that G is isomorphic to the direct product $M \times N$. Conclude that G is abelian.

Problem 4

Consider the field $F = \mathbb{Z}_3$ and let $f(x) = x^3 + 2x + 1 \in F[x]$.

4a

Explain why $K = F[x]/\langle f(x) \rangle$ is a field.

4b

We assume known that $K = F(\alpha)$, for $\alpha = x + \langle f(x) \rangle$ in K. Use α to write a basis for K over F. Express α^6 and α^4 in this basis.

4c

Find a monic polynomial $g(x)$ of degree 3 in $F[x]$ such that α^2 is a zero of $g(x)$.

4d

Show that $f(1+\alpha) = 0$ and $f(2+\alpha) = 0$. Conclude that K is a splitting field of $f(x)$ over F.

Problem 5

Let $g(x) = x^3 + 4$ and $f(x) = x^5 + 2x^4 + 3x^3 + 4x^2 + 8x + 12$ in $\mathbb{Q}[x]$. Let L be the splitting field of $g(x)$ over $\mathbb{Q}$ and K the splitting field of $f(x)$ over $\mathbb{Q}$.

5a

Show that $g(x)$ divides $f(x)$ in $\mathbb{Q}[x]$ and that $f(x)$ has no zeros in $\mathbb{Q}$.

5b

Find the splitting field L and determine the degree $[L : \mathbb{Q}]$.

(Continued on page 4.)

5c

Prove that $[K : L] = 2$ and determine $[K : \mathbb{Q}]$. Explain why K is a finite normal extension of $\mathbb{Q}$.

5d

Explain why $G(K/L)$ is a normal subgroup of the Galois group $G(K/\mathbb{Q})$. Show that $G(K/\mathbb{Q})$ contains a normal subgroup N of order 6, and that $G(K/\mathbb{Q})$ is the direct product of N and $G(K/L)$.

SLUTT.