1. Since $(g^{-1})^{-1} = g$, the map is bijective. It is a homomorphism if $a^{-1}b^{-1} = (ab)^{-1}$ for all $a, b \in G$. Since $(ab)^{-1} = b^{-1}a^{-1}$, and a^{-1} and b^{-1} can be arbitrary elements of the group, this is the same as to require $xy = yx$ for all $x, y \in G$.

2. Every element in S_4 different from the identity is a product of disjoint cycles of length ≥ 2. It is of order two if the cycles are of length 2. Therefore elements of order 2 are cycles of length 2, that is, transpositions, and products of two disjoint transpositions. There are $\binom{4}{2} = 6$ transpositions. Since for a given transposition there is exactly one disjoint transposition, there also are 6 ordered pairs of disjoint transpositions, and hence $6/2 = 3$ unordered pairs of disjoint transpositions. Thus we have $6 + 3 = 9$ elements of order 2.

3. (a) Any cycle of length n can be uniquely written as $(i_1, \ldots, i_{n-1}, n)$, where $i_1, \ldots, i_{n-1}$ are different elements of $\{1, 2, \ldots, n-1\}$. Hence the number of such cycles is $(n-1)!$.

(b) The nontrivial orbit of a cycle of length k, $2 \leq k \leq n$, is a k-element subset of $\{1, \ldots, n\}$. There are $\binom{n}{k}$ such subsets. By (a) the number of cycles of length k with a given orbit is $(k-1)!$. Hence the total number of cycles of length k is

$$\binom{n}{k}(k-1)! = \frac{n(n-1)\ldots(n-k+1)}{k}.$$

4. It is easy to check that every nonzero scalar matrix is in the center. Conversely, assume $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is in the center. Since

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} a & -b \\ c & -d \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ -c & -d \end{pmatrix},$$

we see that $b = c = 0$. Since

$$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a \\ d & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} = \begin{pmatrix} 0 & d \\ a & 0 \end{pmatrix},$$

we see that $a = d$, as required.

Many students used in one or another form that if $A \in \mathrm{GL}(2, \mathbb{R})$ is in the center then $AB = BA$ for any, not necessarily invertible, matrix B, forgetting to explain why this is the case. One possible explanation is that any matrix is a sum of two invertible matrices. Indeed, if $B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$ then take any λ different from $0, e, h$ and write

$$B = \begin{pmatrix} e-\lambda & f \\ 0 & h-\lambda \end{pmatrix} + \begin{pmatrix} \lambda & 0 \\ g & \lambda \end{pmatrix}.$$

5. (a) Since an automorphism is a bijective map of a group onto itself, $\mathrm{Aut}(G)$ is a subset of the permutation group S_G. Since the inverse of an isomorphism is an isomorphism, and the composition of two isomorphisms is an isomorphism, $\mathrm{Aut}(G)$ is a subgroup.

(b) Since $\iota_g\iota_{g^{-1}} = \iota_{g^{-1}}\iota_g = \mathrm{id}$, ι_g is bijective. Since

$$\iota_g(xy) = gxyg^{-1} = gxg^{-1}gyg^{-1} = \iota_g(x)\iota_g(y),$$

ι_g is an automorphism.

(c) We have $\iota_g\iota_h = \iota_{gh}$ and $\iota_g^{-1} = \iota_{g^{-1}}$, so $\mathrm{Inn}(G)$ is a subgroup of $\mathrm{Aut}(G)$. If $\alpha \in \mathrm{Aut}(G)$ then

$$(\alpha\iota_g\alpha^{-1})(h) = \alpha(g\alpha^{-1}(h)g^{-1}) = \alpha(g)h\alpha(g)^{-1},$$

so that $\alpha\iota_g\alpha^{-1} = \iota_{\alpha(g)} \in \mathrm{Inn}(G)$. Thus $\mathrm{Inn}(G)$ is normal in $\mathrm{Aut}(G)$.

6. Since any automorphism leaves the identity element invariant, we can consider $\mathrm{Aut}(G)$ as a subgroup of the permutation group $S_{G\setminus\{e\}} \cong \{\sigma \in S_G \,|\, \sigma(e) = e\}$. Let $G\setminus\{e\} = \{a, b, c\}$. Then the multiplication in our group G is described by the following rules: the square of every element in $\{a, b, c\}$ is e, and the product of any two different elements in $\{a, b, c\}$ is the third element. It is clear that for any permutation $\sigma \in S_{G\setminus\{e\}}$ the elements $\sigma(a), \sigma(b), \sigma(c)$ satisfy the same rules, so σ defines an automorphism of G. Thus $\mathrm{Aut}(G) \cong S_{G\setminus\{e\}} \cong S_3$.

7. (a) Elements $h_1, h_2 \in H$ represent the same left coset of K in G, that is, $h_2^{-1}h_1 \in K$, if and only if they represent the same left coset of $H \cap K$ in H, that is, $h_2^{-1}h_1 \in H \cap K$. Hence the number of left cosets of K in G is not smaller than the number of left cosets of $H \cap K$ in H. In other words, $(G\colon K) \geq (H\colon H \cap K)$.

Notice that by the above argument we can identify the set $H/(H \cap K)$ of left cosets of $H \cap K$ in H with a subset of the set G/K of left cosets of K in G. In terms of group actions this can be done as follows, which gives an alternative solution of the exercise. Consider the action of H on G/K. Then the isotropy group of $K \in G/K$ is $H \cap K$, and hence $H/(H \cap K)$ can be identified with the H-orbit of $K \in G/K$ in G/K.

(b) Since $(H\colon H \cap K)$ is finite by (a), $(G\colon H \cap K) = (G\colon H)(H\colon H \cap K)$ is also finite.

Here is another argument. Observe that $g(H \cap K) = gH \cap gK$. Hence the number of left cosets of $H \cap K$ is not larger than (in fact, equal to) the number of nonempty intersections of left cosets of H with left cosets of K. Thus $(G\colon H \cap K) \leq (G\colon H)(G\colon K)$, since the latter product is the number of all possible intersections.

Yet another possibility is to consider the action $g(x, y) = (gx, gy)$ of G on the finite set $(G/H) \times (G/K)$ and notice that $H \cap K$ is the isotropy group of the point (H, K).

8. (a) We have $hgx = gx$ if and only if $g^{-1}hgx = x$. That is, $g^{-1}G_{gx}g = G_x$. Equivalently, $G_{gx} = gG_xg^{-1}$.

(b) Since the action is transitive, we have only one orbit. By Burnside's formula,

$$|G| = \sum_{g\in G} |X_g|.$$

Since $|X_e| = |X| \geq 2$, we cannot have $|X_g| \geq 1$ for all g. So there is g such that $X_g = \emptyset$, that is, $gx \neq x$ for all $x \in X$.

(c) Consider the transitive action of G on the set $X = G/H$ of left cosets of H. The isotropy group of $H \in X$ is H itself. By (a) we conclude that $\cup_{g\in G} gHg^{-1}$ is the union of the isotropy groups of all points. But by (b) there exists an element g which does not belong to any isotropy group.

Several students gave the following argument. Consider the set $\{g_1Hg_1^{-1}, \ldots, g_nHg_n^{-1}\}$ of all conjugates of H. The group G acts on it transitively by conjugation. The isotropy

group of H is the normalizer $N(H)$ of H in G. Hence $n = (G\colon N(H))$. Since $H \subset N(H)$, we conclude that $n \le (G\colon H)$. But then

$$\left|\cup_{i=1}^{n} g_i H g_i^{-1}\right| \le 1 + \sum_{i=1}^{n} |g_i H g_i^{-1} \setminus \{e\}| \le 1 + (G\colon H)(|H| - 1) = 1 + |G| - (G\colon H) < |G|.$$

(d) We have to find an example of an infinite group G with a proper subgroup H such that $G = \cup_{g\in G} gHg^{-1}$. The latter condition means that for every $a \in G$ there exists $g \in G$ such that $gag^{-1} \in H$. Consider two such examples.

(i) Let X be an infinite set. Let G be the group of finite permutations of X, that is, a permutation $\sigma \in S_X$ belongs to G if there exists a finite subset $F \subset X$ such that $\sigma(x) = x$ for $x \notin F$ (so σ permutes only a finite number of points). Let Y be an infinite subset of X, $Y \ne X$, and $H \subset G$ consists of permutations $\sigma \in G$ such that $\sigma(x) = x$ for $x \notin Y$.

Let $\sigma \in G$, so there exists a finite subset $F \subset X$ such that $\sigma(x) = x$ for $x \notin F$. Since Y is infinite, we can choose a subset $F' \subset Y$ with $|F'| = |F|$. Take a permutation $g \in G$ such that $g(F) = F'$. Then if $x \notin F'$, we have $g^{-1}(x) \notin F$, so that $\sigma(g^{-1}(x)) = g^{-1}(x)$, and hence $g(\sigma(g^{-1}(x))) = x$. Thus $g\sigma g^{-1} \in H$.

(ii) Let $n \ge 2$. Consider the group $G = \mathrm{GL}(n, \mathbb{C})$ of invertible n by n matrices with complex coefficients. Let H be the subgroup of upper triangular matrices. From the linear algebra course you should know that any matrix can be written in the normal Jordan form, in particular, in the upper triangular form. This means exactly that for any matrix B there exists $A \in \mathrm{GL}(n, \mathbb{C})$ such that ABA^{-1} is upper triangular.

For $n = 2$ the same example can be considered from the following more geometric point of view, used by some students. Consider the extended complex plane $X = \mathbb{C} \cup \{\infty\}$. Let G be the group of fractional linear transformations, so G consists of maps $T\colon X \to X$ of the form $T(z) = \dfrac{az + b}{cz + d}$, where $a, b, c, d \in \mathbb{C}$, $ad - bc \ne 0$. Then we take H to be the subgroup of linear transformations, $T(z) = az + b$, $a \ne 0$. Notice that the action of G on X is transitive and H is the isotropy group of the point ∞, so the equality $G = \cup_{g\in G} gHg^{-1}$ is equivalent to the fact that any fractional linear transformation has a fixed point.

The above examples do not work with real matrices for the simple reason that any upper triangular real matrix has only real eigenvalues. It is possible, however, to get required examples with real matrices using larger subgroups than the group of upper triangular matrices. E.g. we can take $G = \mathrm{GL}(3, \mathbb{R})$ and the subgroup H consisting of matrices of the form

$$\begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix}.$$

For the proof use either Schur factorization or the fact that any real 3 by 3 matrix has at least one real eigenvalue.