SOLUTIONS TO FINAL EXAM MAT2200 SPRING 2008

Problem 1

a: All abelian gruops are of the form

$$Z_{p_1^{r1}} \times \ldots \times Z_{p_s^{rs}}$$

so the ones of order 8 are

$$Z_8, Z_4 \times Z_2, Z_2 \times Z_2 \times Z_2$$

b: The symmetry group of the square, D_4, has order 8 and is not abelian.

c: By Sylows theorem there are 1 or 3 Sylow 2-subgroups of S_4. The symmetry group of the square is a subgroup, since it permutes the 4 sides. We may label the sides with a, b, c, d such that a, b ar opposite, a, c are opposite or a, d are opposite. These three ways of labeling the sides identifies the symmetri group of the square as a subgroup of S_4 in three different ways, so S_4 has three subgroups of order 8, all isomorphic to the nonabelian group D_4.

Problem 2

a: $x^5 + 4x^3 + 2$ is irreducible over Q by Eisensteins criterion (2|4, 2|2, but 4 does not divide 2).

b: A quadratic polynomial is irreducible over Z_3, if and only if it has no zero in Z_3. Of the possible (monic) polynomials

$x^2, x^2 + x, x^2 - x, x^2 + 1, x^2 - 1, x^2 + x + 1, x^2 + x - 1, x^2 - x + 1, x^2 - x - 1$

all but $x^2 + 1, x^2 + x - 1, x^2 - x - 1$ has a zero. Likewise, $x^5 + 4x^3 + 2$ has a zero at 2, so it is reducible.

Problem 3

a: $x^4 - 2x^2 - 3 = (x^2 - 3)(x^2 + 1)$ so the splitting field over Q is $E = Q(\sqrt{3}, i)$. Furthermore $[E : Q] = 4$ and the Galois group G is $Z_2 \times Z_2$ (it has order 4 and both $G(E/Q(i))$ and $G(E/Q(\sqrt{3}))$ are subgroups of order 2, so it has at least two subgroups of order 2).

b: $i+\sqrt{3} \in E$ and has minimal polynomial x^4-4x^2+16, so $E = Q(i+\sqrt{3})$.