UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in: MAT2200 — Groups, Rings and Fields

Day of examination: 14. June 2016

Examination hours: 14.30 – 18.30

This problem set consists of 5 pages.

Appendices: none

Permitted aids: none

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

All problems have equal weight.

Problem 1

1a

Find all abelian groups of order 12 up to isomorphism.

Solution: $12 = 3 \cdot 2 \cdot 2 = 3 \cdot 4$ are the different ways of factoring 12 into powers of primes, so an abelian group of order 12 is $\mathbb{Z}_3 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ or $\mathbb{Z}_3 \oplus \mathbb{Z}_4$

1b

Show that the matrices

$$M = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \quad | \quad a, c \in \mathbb{Z}_3 \setminus \{0\},\ b \in \mathbb{Z}_3 \right\}$$

form a group under matrix multiplication.

Solution: $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \cdot \begin{pmatrix} a' & b' \\ 0 & c' \end{pmatrix} = \begin{pmatrix} aa' & ab' + bc' \\ 0 & cc' \end{pmatrix}$ and $aa', cc' \in \mathbb{Z}_3 \setminus \{0\}$ when $a, a', c, c' \in \mathbb{Z}_3 \setminus \{0\}$, so M is closed under matrix multiplication. Matrix

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multiplication is associative, contains $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} a^{-1} & -bc^{-1}a^{-1} \\ 0 & c^{-1} \end{pmatrix}$, the inverse of $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$, so M is a group.

What is the order of M? Find and list all elements in M of order 3. Is M abelian? Give reasons for your answer.

Solution: $|M| = 2 \cdot 2 \cdot 3 = 12$, since there are two choices for a and c, and three for b. If $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ has order 3, then $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}^3$ has $a^3 = c^3 = 1$ on the diagonal, so $a = c = 1$. Furthermore $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}^3 = \begin{pmatrix} 1 & b+b+b \\ 0 & 1 \end{pmatrix}^3$, and $b + b + b = 0$ for any $b \in \mathbb{Z}_3$, so these are the two elements of order 3 in M:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}.$$

Since $\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \neq \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 0 & 2 \end{pmatrix}$ the group M is not abelian.

1c

A permutation is even (resp. odd) if it is a product of an even (resp. odd) number of transpositions.
Show that a 3-cycle is even, and that a 4-cycle is odd.

Solution: $(123) = (13)(12)$ so a 3-cycle is even. $(1234) = (14)(13)(12)$ so a 4 -cycle is odd.

1d

The set of even permutations on 4 elements is a group of order 12. Is it isomorphic to M? Give reasons for your answer.

Solution: Every 3-cycle is even so it belongs to the group A_4 of even permutations on $\{1, 2, 3, 4\}$. A 3-cycle has order 3, so A_4 contains more than 2 3-cycles, for example $(1, 2, 3)$, $(1, 2, 4)$ and $(2, 3, 4)$, while M contains only 2, so they are not isomorphic.

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Problem 2

Let K be the splitting field of $f = (x^3 - 8)(x^2 - 2) \in \mathbb{Q}[x]$.

2a

Find the degree $[K : \mathbb{Q}]$ and the group $G(K/\mathbb{Q})$.

Solution: $f = (x^3 - 8)(x^2 - 2) = (x - 2)(x^2 + 2x + 4)(x^2 - 2) \in \mathbb{Q}[x]$, and both $(x^2 + 2x + 4)$ and $(x^2 - 2)$ are irreducible over $\mathbb{Q}$. The splitting field of $(x^2 - 2)$ is $\mathbb{Q}(\sqrt{2})$, while the splitting field of $(x^2 + 2x + 4)$ is $\mathbb{Q}(i\sqrt{3})$, so $K = \mathbb{Q}(\sqrt{2}, i\sqrt{3})$. Both extensions have order 2, so $[K : \mathbb{Q}] = 4$ and $G(K/\mathbb{Q}) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

2b

Find an element $a \in K$ such that $K = \mathbb{Q}(a)$. **Solution:** Let $a = \sqrt{2} + i\sqrt{3}$,

then $a^2 = -1 + i\sqrt{6}$. $1, a$ and a^2 are linearly independent over $\mathbb{Q}$, so the minimal polynomial of a cannot have degree 2, so it must have degree 4 (since the degree must divide 4.) Therefore $K = \mathbb{Q}(a)$.

Problem 3

3a

What is an integral domain?

Solution: An integral domain is a commutative ring such that if the product of two elements in the ring is 0, then one of the elements is 0.

Show that $\mathbb{Z}[x]/(x^2 + 1)$ is a integral domain.

Solution: $x^2 + 1$ is irreducible over $\mathbb{Z}$, so it generates a prime ideal in $\mathbb{Z}[x]$. The quotient of a commutative ring by a prime ideal is an integral domain, so $\mathbb{Z}[x]/(x^2 + 1)$ is an integral domain.

Is $\mathbb{Z}[x]/(x^2 - 1)$ an integral domain? Give reasons for your answer.

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Solution: Neither $x+1$ nor $x-1$ belongs to the ideal generated by x^2-1 in $\mathbb{Z}[x]$, while their product do, so their residue classes are nonzero and have a product is zero in $\mathbb{Z}[x]/(x^2-1)$, so the latter ring is not an integral domain.

Problem 4

An element $r \neq 0$ in a ring is nilpotent if $r^n = 0$ for some $n > 1$.

4a

Show that the set of nilpotent elements in a commutative ring together with 0 form an ideal.

Solution: Let $N \subset R$ be the set of nilpotent elements and 0 in the ring R. If $a \in N$ and $b \in N$, then $a^n = 0$ for some n and $b^m = 0$ for some m. Then $(a+b)^{n+m} = a^{n+m} + ... + b^{n+m} = 0$, so $a+b \in N$. Also $(ra)^n = r^n a^n = 0$, so N is closed under addition and multiplication by elements in R and contains 0 so it is an ideal.

4b

The set of matrices

$$M' = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad | \quad a,b,c,d \in \mathbb{Z}_3 \right\}$$

with matrix addition and matrix multiplication is a ring.

Show that a nilpotent element in M' has determinant $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = 0$.

Solution: If $A \in M'$ is nilpotent, then $A^n = 0$ for some n. But $\det A^n = (\det A)^n = 0$. Since $\mathbb{Z}_3$ is an integral domain, this means that $\det A = 0$.

Find and list all nilpotent elements in M'.

Solution: We first list all matrices $A \in M'$ of determinant 0:

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$$

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$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 2 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}$$

of these, taking powers of these matrices, one gets that only the following are nilpotent:

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}$$

END