Mandatory assignment in MAT2200 Spring 2010

The solution to the assignment must be delivered in the department office B714 on the 7th floor in the Niels Henrik Abel building before 14:30 on Friday March 19.

Note: You must provide proofs for all your answers. All individual questions carry the same weight. To pass the assignment you will need a score of at least 50%.

Exercise 1. (The questions (a)-(c) in this exercise are not related to each other.)

(a) Compute the number of generators of $\mathbb{Z}_5$. To which known group is $\mathrm{Aut}(\mathbb{Z}_5)$ isomorphic to?

(b) Find, up to isomorphism, all finite abelian groups of order 96.

(c) Find two subgroups of S_4 which are isomorphic to S_3 (there are four such subgroups).

Exercise 2. Let G and G' be abelian groups. Let $\mathrm{Hom}(G, G')$ denote the set of all homomorphisms from G to G'. Define an operation $+$ on $\mathrm{Hom}(G, G')$ as follows: for $f, g \in \mathrm{Hom}(G, G')$ let

$$f + g : G \to G', (f + g)(x) = f(x) + g(x) \text{ for all } x \in G. \tag{1}$$

(a) Show that $f + g \in \mathrm{Hom}(G, G')$. Explain in the proof where the assumption that G' is abelian is used.

(b) Show that $(\mathrm{Hom}(G, G'), +)$ is an abelian group.

(c) Suppose $G = \mathbb{Z}$. Show that $\mathrm{Hom}(\mathbb{Z}, G')$ is isomorphic to G'. (Hint: use the evaluation $\psi(f) = f(1)$ for $f \in \mathrm{Hom}(\mathbb{Z}, G')$.)

(d) Describe the elements in $\mathrm{Hom}(\mathbb{Z}, \mathbb{Z})$.

Exercise 3. Let $G = \mathbb{Z} \times \mathbb{Z}$.

(a) Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M(2, \mathbb{Z})$, and assume that $\det(A) = \pm 1$. Show that the map $\phi_A : G \to G$ given by $\phi_A(x, y) = (ax + by, cx + dy)$ is an automorphism of G.

(b) Suppose that $\phi : G \to G$ is a homomorphism. Show that there is $A \in M(2, \mathbb{Z})$ such that $\phi = \phi_A$.

(c) If ϕ is an automorphism of G show that $\phi = \phi_A$ where $\det(A) = \pm 1$. (Hint: use Cramer's rule).

Exercise 4. Let G be a group and H a subgroup of G.

(a) Let X be the collection of all left cosets of H. Let G act on X by left translation $g(xH) = gxH$ for $g \in G$ and $xH \in X$. Let $\phi : G \to S_X$ be the homomorphism given by $\phi(g) = \sigma_g$ for $g \in G$, where $\sigma_g(xH) = gxH$ for all $xH \in X$. Show that $\ker \phi$ is contained in H.

(b) Assume that G has order pn where p is a prime such that $p > n$, and assume H has order p. Show that H is normal in G.