The symmetries of a square — a motivation.

Warning: This note was originally written in Norwegian, and this a very quick and dirty translation into English. May be a better version follows

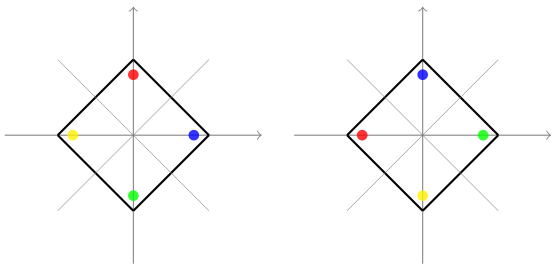
This note is ment as an introduction with aim to motivate the axiomatic introduction of groups. It also gives the first examples of groups, and is a first illustration of how one can "compute" with group elements.

The symmetries of a square

We start with a look at the symmetries of simple and well known figure, namely the square. We imagine the square lying the complex plane with corners in the points $1, i, -1$ and $-i$ as shown in figure 1.

We have an intuitive understanding of what the symmetries of a figure are. One way to make it a little more precise, is to imagine the figure being cut out of wooden a plate or a plate made of some other rigid material. One may then describe a symmetry of the figure as a way of moving it. Lift it out, turn it in some way, and put it back.

There are of course completely precise ways of expressing this mathematically. One is to interpret a symmetry as a mapping ϕ from $\mathbb{C}$ to $\mathbb{C}$ preserving distances — what one usually calls an *isometry* — which carries the square into it self, i.e., $\phi(K) = K$, K being the square.



Figur 1: *A square with corners* $1, i, -1$, *and* -1 *and its lines of symmetry. To the right the square after being rotated* $90°$.

One recognizes immediately several symmetries of the square, the first ones being the rotations. One may turn the square an angle $90°$ about an axis centered in the origin and orthogonal to the plane. One may turn it 180° or 270°. This gives three rotations, all three being symmetries.

It is appropriate to make it precise that in this context what matters are the results of the manipulations of the square, how it is placed after being put back. It is irrelevant what happens to it in the meantime. If we trow it in the air or place it in a drawer for some time, it doesn't matter. Two symmetries are considered to be equal, if in the two cases the square is identically placed after having been put back.

With this clarification, almost all other rotations of the square will be identical to one of the three we described above. For example, then rotation of the square by $-90°$ is identical to a rotation by 279°. However, there are some exceptions. After being turned 360° and put back, the square will be in the same position. Of course the same is true for any rotation by a multiple of 360°, be it positive or negative.

We shall regard this as asymmetry as well, and shall call it the *trivial symmetry*, or the *identity*.

The square has *four* lines of symmetry, indicated in gray in figure 1.There are the two axis and the two lines $y = x$ and $y = -x$. If we turn the square 180° about one of these lines, we get a symmetry. This gives four new symmetries.

These four together with the identity and the three rotations are all the symmetries of the square, so it adds up to eight. We denote this group of symmetries by D_8.

To be able to decide if a square is in the same position when put back, one needs some kind of marking of the corners. We have used coloured dots in the figures.

One symmetry operation may follow another, and of course the final result may be considered as a symmetry — we have lifted out the square, manipulated it and finally put it back. The resulting symmetry is called the *composition* of the two. It is denoted $\beta\alpha$ if α is the first symmetry and β the second to be performed.

As the trivial symmetry does not do anything, composing any other symmetry with it, doesn't change anything. This justifies the notation 1 for the trivial one, we then have the identity $\alpha 1 = 1\alpha = \alpha$.

Every symmetry has what we call an *inverse* symmetry. That is the symmetry which neutralizes the first one , and it is easy to imagine how it works: Just lift out the square and put it back like was at the beginning! The inverse symmetry to α is denoted by α^{-1}. In symbols we have $\alpha^{-1}\alpha = 1$. Correspondingly $\alpha\alpha^{-1} = 1$ — with a little afterthought we realize that α neutralizes α^{-1}.

We shall now introduce symbols for the rotations, and let r_i denote the rotation with $i \cdot 90°$. Then r_1, r_2 and r_3 are the three rotations we mentioned above. We have that $r_3 = r_1^{-1}$, and r_2 is its own inverse!

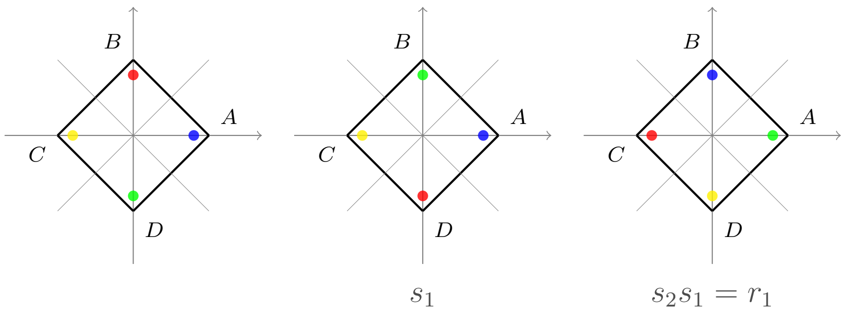
Clearly $r_2 = r_1^2$ since two rotations in a row, each of 90°, compose to one of 180°. Likewise $r_3 = r_1^3$, since three turns of 90° give one of 270°.

Simplifying the notation, we write r for r_1 (dropping the subscript) and r^i with the appropriate i for all the other rotations. Only r^2 and r^3, or perhaps r^{-1}, will be needed, since every other equals one of these.

Composition of symmetries

To fully understand the symmetries of the square we must understand how they compose — or what amounts to the same, understand the product of the different symmetries.

To keep track of things, we also mark the corners of the wooden plate with the letters A, B, C an D as in figure 2.



Figur 2: *Kvadratet i utgangsposisjon til venstre.*

We have already taken a look at the composition of the different rotations. So let us proceed by studying the composition of some reflections. The reflection about the x-axis will be denoted by s_1 and the one about the line $y = x$ by s_2.

The figure 2 shows the square in is original position, after having been acted upon by s_1 and by the composition $s_1 s_2$. On comparing with figure 1, one recognizes the result. It is nothing but the rotation r!

We want a somehow more formal approach than just looking at a figure. As we said, we shall follow the corners, and to do that, we introduce the notation $s : X \mapsto Y$. It means that the symmetry s send corner of the square placed at X tot Y. (X and Y are among the "corners" A, B, C and D of the wooden plat).

The rotation r will in this notation be

$$r_1 : A \mapsto B \mapsto C \mapsto D \mapsto A.$$

The reflection s_1 — which was about the x-axis — is expressed as

$$s_1 : B \mapsto D \mapsto B,$$

which means that at corners located at B and D are interchanged, while the corners at A and C do not move — with the convention that corners that do not move, are not mentioned.

The reflection s_2 — about the line $y = x$ — is written as

$$s_2 : A \mapsto B \mapsto A \text{ and } C \mapsto D \mapsto C$$

Let us now follow the corners when we use s_1 followed by s_2 :

$$A \xmapsto{s_1} A \xmapsto{s_2} B \xmapsto{s_1} D \xmapsto{s_2} C \xmapsto{s_1} C \xmapsto{s_2} D \xmapsto{s_1} B \xmapsto{s_2} A$$

The total effect is $A \mapsto B \mapsto C \mapsto D \mapsto A$, which is r! We have:

$$s_2 s_1 = r. \tag{1}$$

What about the composition the other way around, i.e., $s_1 s_2$? We leave the precise tracking of corners to the reader, but it is illustrated on figure 3. The result is

$$s_1 s_2 = r^{-1}. \tag{2}$$

We could also have found this out just by thinking! Clearly $s_1^2 = s_2^2 = 1$. Using that we get

$$(s_2 s_1)(s_1 s_2) = s_2(s_1 s_1)s_2 = s_2 1 s_2 = s_2^2 = 1,$$

and from that, by applying $(s_2 s_1)^{-1}$ på to both sides of that equality, we get $s_1 s_2 = (s_2 s_1)^{-1} = r^{-1}$.

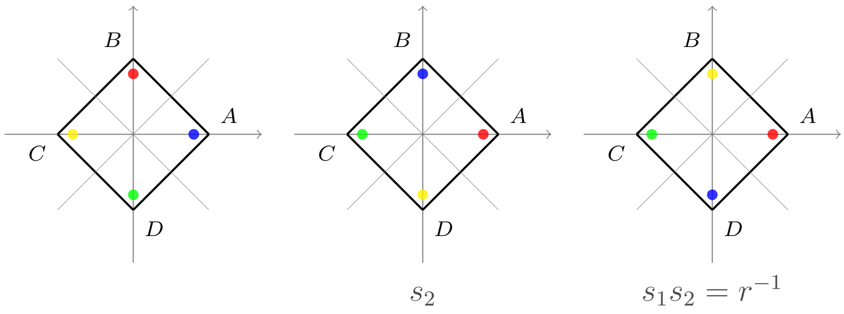
The first comment is that $s_1 s_2 \neq s_2 s_1$ — so in the groups the order of the factors matters! They do not necessarily *commute*, as we say.

The second comment is that that we now have a full knowledge of the symmetry group of the square! One may express any symmetry either as compositions of s_1, s_2 or by compositions of s_1 and r, And to do “computations” with the symmetries, the only rules we need in addition to the usual ones, are $s_1^2 = s_2^2 = 1$ and $r^4 = 1$.

In what follows, we are going to explain this. Recall that $r = s_2 s_1$.

An important rule is:

$$s_1 r s_1 = r^{-1}$$



Figur 3: *Kvadratet i utgangsposisjon til venstre*

which follows since $s_1 r s_1 = s_1(s_2 s_1)s_1 = (s_1 s_2)(s_1 s_1) = r^{-1}$, as $s_1^2 = 1$ and $s_1 s_2 = r^{-1}$. With a rule like that, we may replace the simple interchanging of the order of factors by a little more complicated rule:

$$s_1 r = r^{-1} s_1 \qquad (\star)$$

so we can (in the case of r and s_1) interchange the factors provided we invert r! A similar rule is valid for the product of any reflection and any power of r. (Check it for s_2 and r)

All the eight symmetries of the square are expressed as compositions of s_1 and r the following way:

The identity and three rotations as powers of r: 1, r, r^2 and $r^{-1} = r^3$. We have s_1 and $s_2 = r s_1$, and the two remaining reflections are $r^{-1}s_1$ and $r^2 s_1$.

Let us check that the square of $r^{-1}s_1$ equals 1:

$$(r^{-1}s_1)(r^{-1}s_1) \overset{(\star)}{=} (s_1 r)(r^{-1}s_1) = s_1(rr^{-1})s_1 = s_1 s_1 = 1,$$

where we use equation $(\star)$ and that $s_1^2 = 1$. A similar computation can be done for $(r^2 s_1)^2$. This means that both $r^{-1}s_1$ and $r^2 s_1$ are reflections, but which one is about the y-axis and which one about the line $y = -x$, is so far not clear — we leave that to the reader.

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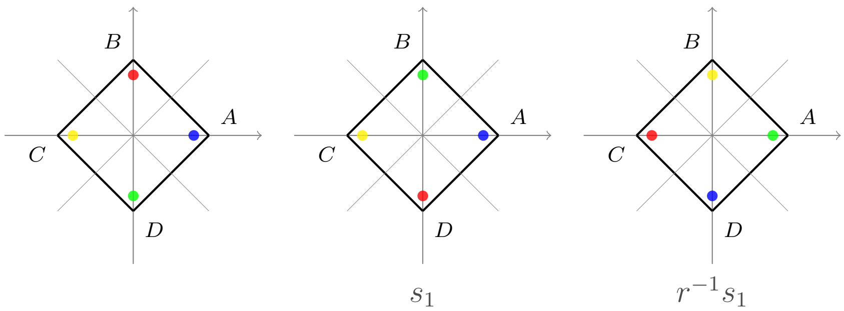
1. Check that that the square of $r^2 s_1$ is 1.

2. Let s be one of the reflections. Show that $sr = r^{-1}s$,and that $sr^i = r^{-i}s$ (Use induction on i).

3. Show by tracing the corners that $s_1 s_2 = r^{-1}$.

4. Express all the eight symmetries by s_1 and s_2.
5. Show that r^2 commutes with all the other symmetries.
6. Which symmetries are equal to theirs own inverse?
7. What is the axis of symmetry for $r^{-1}s_1$. And for $r^2 s_1$?



Figur 4: *Figuren viser at $r^{-1}s_1$ har $y = -x$ som symmetriakse*