Exercise 1. Find and classify all groups of order 8.

Solution. By Lagrange theorem, the elements of $G$ have order 1, 2, 4 or 8. If every element has order 2, then by the lemma $G$ is abelian, and we have $G \simeq \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. If there is an element of order 8, the group is cyclic, hence abelian, and $G \simeq \mathbb{Z}_8$. Suppose there is an element $g \in G$ of order $|g| = 4$. Let $h \notin \langle g \rangle$. Then $|h| = 2$. Then as a set $G = \langle g \rangle \times \langle h \rangle$. The structure of $G$ is decided by the product $gh$. Since $h \notin \langle g \rangle$ we have $hg \neq e$. If $hg = g^2 h$, then a conjugate element of $g$, $hgh = g^2$ has order 2, but $g$ has order 4 which is impossible, since conjugation does not change the order of an element. Thus we are left with two possibilities, $hgh = gh$ which gives the abelian group $\mathbb{Z}_4 \times \mathbb{Z}_2$, or $hgh = g^3 h$. The last one we recognize as the dihedral group $D_4$.

Exercise 2. The tetrahedron is a regular solid with 4 vertices and 4 triangular faces. The symmetry group is the alternating group $A_4$. Compute the number of different paintings of a tetrahedron with $n$ colours.

Solution. The set $X$ of all paintings of the cube by up to $n$ different colours has $n^4$ elements. The number of different paintings is the number of orbits in $X$ under the action of the symmetry group of the cube. We shall use the Burnside Formula:

$$r = \frac{1}{|G|} \sum_{g \in G} |X_g|$$

where $r$ be the number of orbits in $X$ under the action of $G$.

We need to know the number of elements in the various fix point sets $X_g$, for all $g \in G$. This can be done by inspection for each element in the symmetry group. The results are listed in the table:

<table>
<thead>
<tr>
<th>symmetry</th>
<th>permutation</th>
<th>order</th>
<th>number</th>
<th>$X_g$</th>
</tr>
</thead>
<tbody>
<tr>
<td>identity</td>
<td>$e$</td>
<td>1</td>
<td>1</td>
<td>$n^4$</td>
</tr>
<tr>
<td>vertex-midpoint-rotation</td>
<td>(123),(132),(124),(142) (134),(143),(234),(243)</td>
<td>2</td>
<td>8</td>
<td>$n^2$</td>
</tr>
<tr>
<td>midedge-midedge-rotation</td>
<td>(12)(34),(13)(24), (14)(23)</td>
<td>2</td>
<td>3</td>
<td>$n^2$</td>
</tr>
</tbody>
</table>
Inserting this into the Burnside Formula we get

\[ r = r(n) = \frac{1}{12}(n^4 + 8n^2 + 3n^2) = \frac{n^2(n^2 + 11)}{12} \]

For some small values of \( n \) we get, \( r(1) = 1, r(2) = 5, r(3) = 15, r(4) = 36. \)

**Exercise 3.**

a) Show that the 24 elements of \( SL_2(\mathbb{F}_3) \) are the matrices

\[ U_{a,b} = \begin{pmatrix} a & b \\ -b & 0 \end{pmatrix}, \quad b \neq 0 \quad \text{and} \quad V_{b,c,d} = \begin{pmatrix} d(b + 1) & b \\ c & d \end{pmatrix}, \quad d \neq 0 \]

There are 6 of the first type and 18 of the second type, and the two families are disjoint.

**Solution.** Notice that for \( a \neq 0 \), we have \( a^2 = 1 \). In general we have

\[ \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc = 1 \]

We consider two separate cases, \( d = 0 \) or \( d \neq 0 \). If \( d = 0 \) we have \( bc = -1 \), i.e. \( c = -b \). This is \( U_{a,b} \). If \( d \neq 0 \), then \( a = (1 + bc)d^{-1} = d(1 + bc) \), which is \( V_{b,c,d} \). The numbers are easily computed, in the \( U \) case as \( 2 \cdot 3 = 6 \) and in the \( V \)-case as \( 3^2 \cdot 2 = 18 \).

b) Show that the only element in \( SL_2(\mathbb{F}_3) \) of order 2 is \( -e = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \).

**Solution.** We have

\[ (U_{a,b})^2 = \begin{pmatrix} a^2 - b^2 & ab \\ -ab & -b^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \]

which is impossible since \(-b^2 \neq 1\), and

\[ (V_{b,c,d})^2 = \begin{pmatrix} d^2(1 + bc)^2 + bc & d(b + 1)c \\ d(b - 1)c & bc + d^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \]

If \( b = c = 0 \), then we must have \( d^2 = 1 \). If \( d = 1 \), then \( V_{0,0,1} = Id \). If \( d = -1 \), then \( V_{0,0,-1} = -Id \).

If \( b \neq 0 \), then we must have \( bc - 1 = 0 \). But then \( d^2 = 0 \) which again is impossible.

c) Show that in the \( U \)-family we have order \( |U_{a,b}| = 4 \) if and only if \( a = 0 \), and in the \( V \)-family we have \( |V_{b,c,d}| = 4 \) if and only if \( bc = 1 \). Thus we have 6 elements of order 4 in the group.

**Solution.** If \( |U_{a,b}| = 4 \), then \( ab = 0 \), \( a^2 - b^2 = -1 \) and \(-b^2 = -1 \), which is possible only if \( a = 0 \).

We have \( |V_{b,c,d}| = 4 \) if \( d^2(1 + bc)^2 + bc = -1 \), \( d(b - 1)c = d(b - 1)c = 0 \) and \( bc + d^2 = -1 \). If \( b = c = 0 \), then \( d^2 = -1 \) which is impossible. If \( b \neq 0 \), then \( bc - 1 = 0 \) and \( d^2 = 1 \).

The 6 elements of order 4 are \( U_{0,1}, U_{0,-1}, V_{1,1,1}, V_{-1,-1,1}, V_{1,1,-1}, V_{-1,-1,-1} \).

\[ U_{0,b} = \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}, \quad V_{b,b,d} = \begin{pmatrix} -d & b \\ b & d \end{pmatrix} \]
d) Let \( H = \{e, -e, U_{0,1}, U_{0,-1}, V_{1,1,1}, V_{-1,-1,1}, V_{1,-1,-1}, V_{-1,1,-1}\} \) be the set consisting of the 6 elements of order 4, together with \( \pm e \). Show that \( H \) is a subgroup of \( SL_2(\mathbb{F}_3) \). Use Sylow theory to prove that \( H \) is normal.

**Solution.** We have \( H = \{\pm e, \pm U_{0,1}, \pm V_{1,1,1}, \pm V_{1,-1,1}\} \). \( H \) contains additive inverses and the identity element. It remains to show that \( H \) is closed under operation. The non-trivial elements have order 4, by c) and therefore they have square \(-Id\) by b). Furthermore,

\[
\begin{align*}
U_{0,1} \cdot V_{1,1,-1} &= -V_{1,1,1} \\
U_{0,1} \cdot V_{1,1,1} &= V_{1,1,-1} \\
V_{1,1,-1} \cdot U_{0,1} &= V_{1,1,1} \\
V_{1,1,-1} \cdot U_{0,1} &= -V_{1,1,-1} \\
V_{1,1,1} \cdot U_{0,1} &= -U_{0,1}
\end{align*}
\]

The group \( H \) is a Sylow-2-subgroup. There are 6 elements of order 4 in \( H \), and there are 6 elements of order 4 in the whole group. Since all Sylow-2-subgroups are conjugate, and therefore preserves order of elements, there can not be more than one sylow-2-subgroup, which then has to be normal.

e) The quaternion group \( Q \) has 8 elements, \( Q = \{\pm 1, \pm i, \pm j, \pm k\} \), where \( i^2 = j^2 = k^2 = -1 \), and \( ij = k = -ji, jk = i = -kj \) and \( ki = j = -ik \). We can write \( Q = \langle i, j \mid i^4 = e, ji = i^3j, j^2 = i^2 \rangle \)

Show that \( H \simeq Q \).

**Solution.** By identification, \( i \mapsto U_{0,1}, j \mapsto V_{1,1,1} \) and \( k \mapsto V_{1,1,-1} \).

The factor group \( SL_2(\mathbb{F}_3)/H \) has \( \frac{24}{8} = 3 \) elements. There is only one group of three elements, the cyclic group \( \mathbb{Z}_3 \). Using multiplicative notation, we write \( \mathbb{Z}_3 = \{H, tH, t^2H\} \), where \( t \) can be taken to be any element in \( SL_2(\mathbb{F}_3) \), outside of \( H \).

f) For your choice of \( t \), list the elements of the two cosets \( tH \) and \( t^2H \).

**Solution.** We choose \( t = U_{-1,-1} \), where \( t^3 = Id \). Then we have

\[
\begin{align*}
tH &= \{\pm U_{-1,-1}, \pm V_{-1,0,1}, \pm V_{0,1,1}, \pm V_{1,-1,1}\} \\
t^2H &= \{\pm V_{1,-1,-1}, \pm V_{0,1,-1}, \pm U_{-1,-1}, \pm V_{1,0,1}\}
\end{align*}
\]

g) In exercise e) you identified \( H \) with the quaternion group, i.e. you identified the quaternion elements \( i, j \) and \( k \) with some matrices in \( SL_2(\mathbb{F}_3) \). We know that \( H \) is a normal subgroup of \( SL_2(\mathbb{F}_3) \), thus we have \( tH = Ht \), and we can find elements \( q_1, q_2, q_3 \in Q \simeq H \) such that \( ti = q_1t, tj = q_2t \) and \( tk = q_3t \). Determine these elements in \( Q \).

**Solution.** We have \( q_1 = tit^2, q_2 = tjt^2 \) and \( q_3 = tkt^2 \). This gives \( q_1 = -V_{1,1,1}, \ q_2 = -V_{1,1,-1}, \ q_3 = U_{0,1} \)

h) Conclude that \( SL_2(\mathbb{F}_3) \) is isomorphic to the group generated by \( i, j \) and \( t \) and write up the defining relations. (We can drop \( k \) since \( k = ij \))

**Solution.**

\[
SL_2(\mathbb{F}_3) = \langle i, j, t \mid i^4 = e, ji = i^3j, j^2 = i^2, t^3 = e, ti = j^3t, tj = ij^3t \rangle
\]
Exercise 4.  
In this exercise we shall classify all groups of order 12. Let $G$ be a group of order $|G| = 12$, and let $P \subset G$ be a Sylow-3-subgroup. Then $|P| = 3$, and the index $(G : P) = 4$. Thus there are 4 left cosets of $P$ in $G$, denoted $[G : P] = \{g_1P, g_2P, g_3P, g_4P\}$  
Thus we have $Sym([G : P]) = S_4$. We define a map 
\[ \phi : G \to Sym([G : P]) \]
where $\phi(g)$ is the bijection of $[G : P]$ given by $\phi(g)(gP) = (gg_i)P$.  

a) Show that $\phi$ is a homomorphism. Remember that the group operation in $Sym([G : P])$ is composition of maps.  
Solution. We have 
\[ \phi(gh)(gP) = ghhg_iP = \phi(g)(hg_iP) = \phi(g)(\phi(h)(gP)) = \phi(g) \circ \phi(h)(gP) \]

b) Show that the homomorphism $\phi$ is injective if and only if $P$ is not a normal subgroup of $G$. In that case we can consider $G$ as a subgroup of $Sym([G : P])$.  
Solution. If $P$ is normal, then for any $g \in P$ we have 
\[ \phi(g)(gP) = ggg_iP = ggg_iP = gP = P \]
and $\phi$ is not injective.  
If $P$ is not normal, then $\phi(g) = \phi(h)$ implies $gg_iP = hg_iP$ for all $g_i \in G$, i.e. $g_i^{-1}h^{-1}gg_iP = P$. Thus we have $g_i^{-1}h^{-1}gg_i \in P$, or $h^{-1}g \in gP$ for all $g_i$. But if $P$ is not normal, and has order 3, then $P \cap g^{-1}Pg = \{e\}$, and it follows that $h^{-1}g = e$, or $h = g$ and $\phi$ is injective.

c) In the case where $P$ is not normal in $G$, use Sylow theory to show that there are 8 elements of order 3 in $G$.  
Solution. $P$ is a Sylow-3-subgroup, and since it is not normal, there must be at least 4 conjugate groups of order 3, with at least 8 different elements of order 3. But 7 Sylow-3-subgroups is impossible, since we then would have 14 elements of order 3, in a group of order 12.  

d) Use the fact that there are 8 elements of order 3 in $S_4$, and any permutations of order 3 is even, to conclude that in this case we have $G \cong A_4$. (Hint: $G \cap A_4$ is a subgroup of $G$ and has at least 8 elements)  
Solution. The intersection $G \cap A_4$ is a subgroup of $G$ with at least $8+1$ elements. By Lagrange we have $G \cap A_4 = G$. It follows that $G \subset A_4$, and again by Lagrange $G \cong A_4$.

Next we consider the case where $P$ is a normal subgroup of $G$ of order 3. Let $P = \langle t \rangle$ where $t^3 = e$. Denote by $Q = G/P$ the factor group. We have $|Q| = 4$. Before we continue we need some more terminology. For any group $G$ we define the automorphism group $Aut(G)$ of $G$. It is the subset of $Sym(G)$ of bijective group homomorphisms.  
e) Show that $Aut(P) \cong \mathbb{Z}_2$ for $P = \mathbb{Z}_3$. 

Solution. An automorphism of $P = \{e, t, t^2\}$ must map $e$ to $e$ and $t$ to either $t$ or $t^2$. Denote the map $t \mapsto t^2$ by $\alpha$. Then $\alpha^2 = \text{Id}$.

As a set, we have $G = P \times Q$, i.e. if $Q = \{q_0 = e, q_1, q_2, q_3\}$, then the elements of $G$ can be written

$$G = \{e, t, t^2, q_1, tq_1, t^2q_1, q_2, tq_2, t^2q_2, q_3, tq_3, t^2q_3\}$$

To determine the structure of $G$ we have to decide which elements in this set that correspond to the products $q_jt$. If we know this, we know the multiplication table of $G$, i.e. we know the structure of $G$.

f) Use the fact that $P$ is a normal subgroup of $G$ to show that $g \mapsto gxg^{-1}$ defines a homomorphism $G \rightarrow \text{Aut}(P)$.

Solution. We have $gh \mapsto ghx(gh)^{-1} = ghxh^{-1}g^{-1} = g[ hxh^{-1} ]g^{-1}$ where $hxh^{-1} \in P$.

g) Show that to give the products $q_jt$ is equivalent to define a group homomorphism $\psi : Q \rightarrow \text{Aut}(P)$, where $\psi(q)(t) = qtq^{-1}$ for the generator $t \in P$.

Solution. Since $P$ is normal the product $q_jt$ must equal $pq_j$ for some element $p \in P$. Thus we associate to any $q \in Q$ and $t \in P$ an element $qtq^{-1} \in P$. This is an homomorphism by f) and an automorphism since $|P| = 3$ and $(qtq^{-1})^2 = qt^2q^{-1}$.

h) If $Q \simeq Z_4$ show that there is a unique nontrivial homomorphism $\psi : Q \rightarrow \text{Aut}(P)$ defining a non-abelian structure on $G$. This group is called the dicyclic group $\text{Dic}_3$ or the generalized quaternion group $Q_{12}$.

Solution. If $Q \simeq Z_4$, the only homomorphism $\psi : Q \simeq Z_4 \rightarrow \text{Aut}(P) \simeq Z_2$ is the trivial one.

i) If $Q \simeq Z_2 \times Z_2$, show that there are 3 possible homomorphisms $\psi : Q \rightarrow \text{Aut}(P)$, but the 3 homomorphisms define isomorphic structures on $G$. This is the dihedral group $D_6$, the symmetry group of a hexagon.

Solution. There are 3 homomorphisms $\psi : Q \simeq Z_2 \times Z_2 \rightarrow \text{Aut}(P) \simeq Z_2$. Any two of them differ by an automorphism of $Q$, inducing an automorphism of $G$.

We conclude that in addition to the two abelian groups $Z_{12}$ and $Z_2 \times Z_6$, there are 3 non-abelian groups of order 12, $A_4$, $\text{Dic}_3 \simeq Q_{12}$ and $D_6$. 