5. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 6 & 1 & 5 & 4 & 3 \end{pmatrix}$

7. 2 **9.** ι

11. $\{1, 2, 3, 4, 5, 6\}$ **13.** $\{1, 5\}$

15. $\epsilon, \rho, \rho^2, \rho^3, \phi, \rho\phi, \rho^2\phi, \rho^3\phi$ where their ϕ is our μ_1. This gives our elements in the order $\rho_0, \rho_1, \rho_2, \rho_3, \mu_1, \delta_1, \mu_2, \delta_2$.

17. 24

19. Referring to Table 8.12, we find that $\langle\rho_0\rangle = \{\rho_0\}$, $\langle\rho_1\rangle = \langle\rho_3\rangle = \{\rho_0, \rho_1, \rho_2, \rho_3\}$, $\langle\rho_2\rangle = \{\rho_0, \rho_2\}$, $\langle\mu_1\rangle = \{\rho_0, \mu_1\}$, $\langle\mu_2\rangle = \{\rho_0, \mu_2\}$, $\langle\delta_1\rangle = \{\rho_0, \delta_1\}$, and $\langle\delta_2\rangle = \{\rho_0, \delta_2\}$. These are all the cyclic subgroups. A subgroup containing one of the "turn the square over" permutations μ_1, μ_2, δ_1, or δ_2 and also containing ρ_1 or ρ_3 will describe all positions of the square so it must be the entire group D_4. Checking the line of the table opposite μ_1, we see that the only other elements that can be in a proper subgroup with μ_1 are ρ_2, μ_2, and, of course, ρ_0. We check that $\{\rho_0, \rho_2, \mu_1, \mu_2\}$ is closed under multiplication and is a subgroup. Checking the row of the table opposite μ_2 gives the same subgroup. Checking the rows opposite δ_1 and opposite δ_2 gives the subgroup $\{\rho_0, \rho_2, \delta_1, \delta_2\}$ as the only remaining possibility, using the same reasoning.

21. **a.** These are "elementary permutation matrices," resulting from permuting the rows of the identity matrix. When another matrix A is multiplied on the left by one of these matrices P, the rows of A are permuted in the same fashion that the rows of the 3×3 identity matrix were permuted to obtain P. Because all 6 possible permutations of the three rows are present, we see they will act just like the elements of S_3 in permuting the entries 1, 2, 3 of the given column vector. Thus they form a group because S_3 is a group.

b. The symmetric group S_3.

23. $\mathbb{Z}_2$ **25.** D_4

27. For $\mathbb{Z}_4$, $\lambda_0 = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{pmatrix}$, $\lambda_1 = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 0 \end{pmatrix}$, $\lambda_2 = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 2 & 3 & 0 & 1 \end{pmatrix}$, $\lambda_3 = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 3 & 0 & 1 & 2 \end{pmatrix}$.
The table for the left regular representation is the same as the table for $\mathbb{Z}_4$ with n replaced by λ_n. For S_3, $\rho_0 = \begin{pmatrix} r_0 & r_1 & r_2 & m_1 & m_2 & m_3 \\ r_0 & r_1 & r_2 & m_1 & m_2 & m_3 \end{pmatrix}$, $\rho_1 = \begin{pmatrix} r_0 & r_1 & r_2 & m_1 & m_2 & m_3 \\ r_1 & r_2 & r_0 & m_2 & m_3 & m_1 \end{pmatrix}$, etc., where the bottom row in the permutation ρ_σ consists of the elements of S_3 in the order they appear down the column under σ in Table 8.8. The table for this right regular representation is the same as the table for S_3 with σ replaced by ρ_σ.

31. Not a permutation **33.** Not a permutation

35. **a.** T **c.** T **e.** T **g.** F **i.** F

37. A monoid **41.** No **43.** Yes

SECTION 9

1. $\{1, 2, 5\}, \{3\}, \{4, 6\}$

3. $\{1, 2, 3, 4, 5\}, \{6\}, \{7, 8\}$

5. $\{2n \mid n \in \mathbb{Z}\}, \{2n + 1 \mid n \in \mathbb{Z}\}$

7. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 1 & 3 & 5 & 8 & 6 & 2 & 7 \end{pmatrix}$

9. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 4 & 3 & 7 & 8 & 6 & 2 & 1 \end{pmatrix}$

11. $(1, 3, 4)(2, 6)(5, 8, 7) = (1, 4)(1, 3)(2, 6)(5, 7)(5, 8)$

13. **a.** 4

b. A cycle of length n has order n.

c. σ has order 6; τ has order 4.

d. 6 in Exercises 10 and 11, 8 in Exercise 12.

e. The order of a permutation expressed as a product of disjoint cycles is the least common multiple of the lengths of the cycles.

15. 6 **17.** 30