21st February, 2017

MAT2200

Mandatory assignment

Submission deadline

Thursday 3rd MARCH 2017, 14:30 in the mandatory activity hand-in box, situated on the 7th floor of Niels Henrik Abels hus.

Instructions

You may write your answers either by hand or on a computer (for instance with LaTeX). All submissions must include the following official front page:

http://www.uio.no/english/studies/admin/compulsory-activities/mn-math-obligforside-eng.pdf

It is expected that you give a clear presentation with all necessary explanations. Remember to include all relevant plots and figures. Students who fail the assignment, but have made a genuine effort at solving the exercises, are given a second attempt at revising their answers. All aids, including collaboration, are allowed, but the submission must be written by you and reflect your understanding of the subject. If we doubt that you understand the content you have handed in, we may request that you give an oral account.

In exercises where you are asked to write a computer program, you need to hand in the code along with the rest of the assignment. It is important that the submitted program contains a trial run, so that it is easy to see the result of the code. In order to print the code from one of the Linux machines belonging to the university, move to the folder containing your program and type

```
lpr -P pullprint_manufacturer filename
```

where `filename` is the file you wish to print and `pullprint_manufacturer` is the name of the manufacturer of the printer you wish print from. Common choices are `pullprint_Ricoh` and `pullprint_HP`.

Application for postponed delivery

If you need to apply for a postponement of the submission deadline due to illness or other reasons, you have to contact the Student Administration at the Department of Mathematics (7th floor of Niels Henrik Abels hus, e-mail: studieinfo@math.uio.no) well before the deadline.

All mandatory assignments in this course must be approved in the same semester, before you are allowed to take the final examination.

Complete guidelines about delivery of mandatory assignments:

uio.no/english/studies/admin/compulsory-activities/mn-math-mandatory.html

GOOD LUCK!

Problem 1. Let G be a group with subgroups H and N and assume N is normal throughout this problem.

a) Prove that under these conditions the set $\{hk : h \in H, k \in N\} \subseteq G$ is a subgroup of G. We will denote it HN.

If $HN = G$ and $H \cap N = \{e\}$ then G is called the *semidirect product* of H and N.

b) Show that G is the semidirect product of H and N if and only if every $g \in G$ may be written uniquely as $g = hk$ with $h \in H$ and $k \in N$. (The uniqueness means that if $hk = h'k'$ with $h, h' \in H$ and $k, k' \in N$ then $h = h'$ and $k = k'$.)

c) Let $\iota : H \to G$ be the inclusion and $\gamma : G \to G/N$ the natural surjection $\gamma(g) = gN$. Show that G is the semidirect product of H and N if and only if the composition $\gamma \circ \iota : H \to G/N$ is an isomorphism.

Problem 2. Let p be a prime number and define $\mathbb{Z}_p^* = \mathbb{Z}_p \backslash \{0\} = \{1, 2, \ldots, p-1\}$. Show that $\mathbb{Z}_p^*$ is a group under multiplication modulo p. (HINT: To prove that every element has an inverse use that all the numbers $2, 3, \ldots, p-1$ are relatively prime to p.)

Problem 3. Let p be a prime number. By $\mathbb{Z}_p$ (resp. $\mathbb{Z}_p^*$) we implicitly mean the group under addition (resp. multiplication) modulo p. Let

$$G_p = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \in \mathbb{Z}_p^*,\ b \in \mathbb{Z}_p \right\} .$$

a) Show that G_p is a group under matrix multiplication and find the order of G_p.

b) Let

$$N_p = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in \mathbb{Z}_p \right\} \text{ and } H_p = \left\{ \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} : a \in \mathbb{Z}_p^* \right\} .$$

Check that both are subgroups and that $N_p \simeq \mathbb{Z}_p$ and $H_p \simeq \mathbb{Z}_p^*$. Show that N_p is a normal subgroup. (You can do this directly or consider the map $\varphi : G_p \to \mathbb{Z}_p^*$ given by

$$\varphi \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} = a\,.$$

Prove that it is a homomorphism and compute the kernel.)

c) Prove that G_p is the semidirect product of H_p and N_p.

d) (Optional) Show that the map

$$G_p \times \mathbb{Z}_p \to \mathbb{Z}_p$$

given by

$$(\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}, x) \mapsto ax + b \mod p$$

defines an action of G_p on the underlying set $\mathbb{Z}_p$ and that the induced map to the permutation group, $\rho : G_p \to S_p$, is injective. Show that $G_3 \simeq S_3$. Viewing G_p via ρ as a group of permutations, prove that it has the following property: Every $\sigma \in G_p$ which is not the identity, leaves at most one element of $\mathbb{Z}_p$ fixed.