Solutions to the mandatory assignment in MAT2200, Spring 2018

I'll omit detailed solutions of several problems that everyone solved with more or less the same solution.

Problem 1. (a) *Show that* $\mathrm{Aut}(G)$ *is a subgroup of the permutation group* S_G*. The automorphisms of the form* $g \mapsto hgh^{-1}$ *are denoted by* $\mathrm{Ad}\, h$ *and called* ***inner****. Show that the set* $\mathrm{Inn}(G)$ *of inner automorphisms is a normal subgroup of* $\mathrm{Aut}(G)$*.*

Solution (sketch): $\mathrm{Aut}(G)$ is a subgroup of S_G, since the composition of isomorphisms is an isomorphism and the inverse of an isomorphism is an isomorphism.

The map $\mathrm{Ad}\colon G \to \mathrm{Aut}(G)$, $g \to \mathrm{Ad}\, g$, is a group homomorphism, hence its image $\mathrm{Inn}(G)$ is a subgroup of $\mathrm{Aut}(G)$. This subgroup is normal, since $\alpha \circ (\mathrm{Ad}\, g) \circ \alpha^{-1} = \mathrm{Ad}\, \alpha(g)$ for any $g \in G$ and $\alpha \in \mathrm{Aut}(G)$.

(b) *Consider the group* $K = \mathbb{Z}_2 \times \mathbb{Z}_2$*. Show that* $\mathrm{Aut}(K) \cong S_3$*.*

Solution: We will use the multiplicative notation for the product on K. Any automorphism of K leaves the identity element e invariant and permutes the remaining 3 elements, hence $\mathrm{Aut}(K)$ can be considered as a subgroup of S_3. To prove that it is isomorphic to the entire group S_3 we have to show that every bijection $f\colon K \to K$ such that $f(e) = e$ is a group homomorphism, that is, $f(xy) = f(x)f(y)$ for all $x, y \in K$. The last identity is clearly true if $x = e$ or $y = e$. Since $a^2 = e$ for all a, it is also true if $x = y$. Finally, if x and y are of order 2 and $x \neq y$, then xy is the remaining third element of order 2. That is, $\{xy\} = K \setminus \{e, x, y\}$ and similarly $\{f(x)f(y)\} = K \setminus \{e, f(x), f(y)\}$, hence $f(xy) = f(x)f(y)$.

(c) *Consider the group* $\mathbb{Z}_n$*. Show that every automorphism of* $\mathbb{Z}_n$ *has the form* $m \mapsto km$ *for some* $k \in \mathbb{Z}_n$ *relatively prime to* n*, where* km *denotes the product in the ring* $\mathbb{Z}_n$*.*

Solution: A homomorphism $f\colon \mathbb{Z}_n \to \mathbb{Z}_n$ is completely determined by the image k of the generator 1, namely, $f(m) = mf(1) = km$ for all $m \in \mathbb{Z}_n$. Any $k \in \mathbb{Z}_n$ can appear this way, since the order of k in $\mathbb{Z}_n$ divides n. Such a homomorphism is an isomorphism if and only if k is again a generator of $\mathbb{Z}_n$, which, as we know, is the case if and only if k is relatively prime to n.

Problem 2. *Assume* G *and* H *are groups and we are given a homomorphism* $\alpha\colon H \to \mathrm{Aut}(G)$, $h \mapsto \alpha_h$*. In this case we say that* H *acts by automorphisms on* G*. Then we can introduce a new product on the set* $G \times H$ *by*

$$(g, h)(g', h') = (g\alpha_h(g'), hh').$$

(a) *Show that this way we get a group structure on* $G \times H$*. The group we thus get is called a* ***semidirect product*** *of* G *and* H *and denoted by* $G \rtimes_\alpha H$*, or simply* $G \rtimes H$ *if* α *is clear from the context.*

Solution (sketch): This is a rather straightforward verification of the axioms. A key point is the formula for the inverse: $(g, h)^{-1} = (\alpha_{h^{-1}}(g^{-1}), h^{-1}) = (\alpha_h^{-1}(g)^{-1}), h^{-1})$.

(b) *Semidirect products can be characterized abstractly as follows. Assume* K *is a group with two subgroups* G *and* H *such that*

- G *is normal in* K*;*
- $G \cap H = \{e\}$*;*
- *every element of* K *can be written as* gh *for some* $g \in G$ *and* $h \in H$*, or in other words, every coset of* G *in* K *contains an element of* H*.*

Show that K *is isomorphic to the semidirect group* $G \rtimes H$*, where* H *acts on* G *by the inner automorphisms* $\mathrm{Ad}\, h$ *of* K *restricted to* G*.*

Solution (sketch): We put $\alpha_h(g) = hgh^{-1}$ for $g \in G$ and $h \in H$. Then the map $G \rtimes_\alpha H \to K$, $(g, h) \mapsto gh$, is a surjective homomorphism. This homomorphism is also injective, since if (g, h) is an element of the kernel, then $gh = e$, hence $g = h^{-1}$, and then $g = h = e$, as $G \cap H = \{e\}$.

(c) *Show that the groups* D_6 *and* S_3 *are isomorphic.*

Solution: Apply 2b) to $K = S_3$, $G = A_3 \cong \mathbb{Z}_3$ and $H = \langle(1, 2)\rangle \cong \mathbb{Z}_2$ (since A_3 is a subgroup of index 2 in S_3 and $(1, 2) \notin A_3$, we have $S_3 = A_3 \cup A_3(1, 2)$, so all the assumptions of 2b) are satisfied). Thus, $S_3 \cong G \rtimes H \cong \mathbb{Z}_3 \rtimes_\alpha \mathbb{Z}_2$ for some $\alpha\colon \mathbb{Z}_2 \to \mathrm{Aut}(\mathbb{Z}_3)$. There are only two homomorphisms $\mathbb{Z}_2 \to \mathrm{Aut}(\mathbb{Z}_3) \cong \mathbb{Z}_2$, the trivial one gives the abelian group $\mathbb{Z}_3 \times \mathbb{Z}_2$, the other one gives D_6. As S_3 is nonabelian, we conclude that $S_3 \cong D_6$.

In a more direct way, one can check that the elements $(1,2,3)$ and $(1,2)$ of S_3 satisfy the same relations as the generators of D_6, and then that the homomorphism $D_6 \to S_3$ such that $a \mapsto (1,2,3)$ and $b \mapsto (1,2)$ is an isomorphism.

Another way is to compare the multiplication tables of the two groups.

Yet another proof can be obtained by using that D_{2n} is the symmetry group of a regular n-gon and then observing that the symmetry group of an equilateral triangle is S_3, since any permutation of the three vertices defines such a symmetry.

We can now start classifying groups of order ≤ 15. Recall the following facts from the lectures or the textbook:

Fact 1. *Classification of finite abelian groups: any such nontrivial group is isomorphic to $\mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_n}$ for uniquely determined numbers $m_1 | \ldots | m_n$ ($m_1 \geq 2$).*

Fact 2. *Any group of prime order p is isomorphic to $\mathbb{Z}_p$.*

Fact 3. *Any group of order p^2, where p is a prime number, is abelian, hence, by Fact 1, is isomorphic to $\mathbb{Z}_{p^2}$ or $\mathbb{Z}_p \times \mathbb{Z}_p$.*

Fact 4. *Any group of order pq, where p and q are prime numbers, $p < q$, $q \not\equiv 1 (\mathrm{mod}\, p)$, is isomorphic to $\mathbb{Z}_{pq}$.*

Fact 5. *Any finite p-group has nontrivial center.*

Fact 2 takes care of the orders $1, 2, 3, 5, 7, 11, 13$: the corresponding groups are cyclic.

Fact 3 takes care of the orders 4 and 9: we get the groups $\mathbb{Z}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_9$ and $\mathbb{Z}_3 \times \mathbb{Z}_3$.

Fact 4 allows us to classify the groups of order 15: we get only the group $\mathbb{Z}_{15}$.

The remaining orders are $6, 8, 10, 12, 14$.

Problem 3. *Let us deal with the orders $6, 10, 14$.*

(a) *Assume G is a group of order pq, where p and q are prime numbers, $p < q$. Compute the possible numbers of Sylow subgroups of G and using Problem 2(b) conclude that G is isomorphic to a semidirect product $\mathbb{Z}_q \rtimes \mathbb{Z}_p$.*

Solution: Since by Sylow's third theorem the number N_p of Sylow p-subgroups of a finite group G (such that p divides $|G|$) divides $|G|$ and satisfies $N_p \equiv 1 \bmod p$, in the present case we have $N_q = 1$ and $N_p = 1$ or $N_p = q$, with the last equality possible only when $q \equiv 1 \bmod p$.

Choose a Sylow q-subgroup Q and a Sylow p-subgroup P of G. Since Q is the unique Sylow q-subgroup, it must be normal. Then P acts on Q by the automorphisms α_h defined by $\alpha_h(g) = hgh^{-1}$ ($g \in Q$, $h \in P$). We also have $|Q| = q$, $|P| = p$, so that $Q \cong \mathbb{Z}_q$ and $P \cong \mathbb{Z}_p$, and $P \cap Q = \{e\}$, since $|P \cap Q|$ must divide both $|Q|$ and $|P|$. By the solution of 2b) we then get an injective homomorphism $Q \rtimes_\alpha P \to G$, $(g,h) \mapsto gh$. Since $|Q \rtimes_\alpha P| = qp = |G|$, this homomorphism must also be surjective. Thus, $G \cong Q \rtimes_\alpha P \cong \mathbb{Z}_q \rtimes_\beta \mathbb{Z}_p$ for some homomorphism $\beta \colon \mathbb{Z}_p \to \mathrm{Aut}(\mathbb{Z}_q)$.

We apply this to $p = 2$ and $q = 3, 5, 7$. In order to finish the classification we have to understand the possible actions of $\mathbb{Z}_2$ by automorphisms on $\mathbb{Z}_q$, that is, the automorphisms of order 2 of $\mathbb{Z}_q$.

(b) *Using Problem 1(c) show that the groups $\mathbb{Z}_q$, for $q = 3, 5, 7$, have exactly one automorphism of order 2 each, namely, the one mapping m into $-m$. Conclude that the only (up to isomorphism) groups of order $2q$, with $q = 3, 5, 7$, are $\mathbb{Z}_{2q}$ and D_{2q}.*

Solution: If q is a prime number different from 2, then by 1c) we know that the automorphisms of $\mathbb{Z}_q$ have the form $\alpha(m) = km$, $k \in \mathbb{Z}_q^* = \mathbb{Z}_q \setminus \{0\}$. Then $\alpha^2(m) = k^2 m$. Therefore α has order 2 if and only if $k \neq 1$ and $k^2 = 1$ in $\mathbb{Z}_q$. By now we know from the lectures that since $\mathbb{Z}_q$ is a field, this happens precisely for $k = -1$.[1] But for the solution of the problem it was enough to check that this is the case by directly computing k^2 for all $k \in \mathbb{Z}_q^*$ and $q = 3, 5, 7$.

By 3a) we have $G \cong \mathbb{Z}_q \rtimes_\alpha \mathbb{Z}_2$ for some homomorphism $\alpha \colon \mathbb{Z}_2 \to \mathrm{Aut}(\mathbb{Z}_q)$. By the previous paragraph, there are only two such homomorphisms α. The trivial one gives the group $\mathbb{Z}_q \times \mathbb{Z}_2 \cong \mathbb{Z}_{2q}$, the other one defines D_{2q}.

The remaining orders 8 and 12 are more complicated. Consider first the case of order 8.

[1] Here is basically the same argument: if $k^2 = 1$ in $\mathbb{Z}_q$, then $(k-1)(k+1) = 0$ in $\mathbb{Z}_q$, that is, q divides $(k-1)(k+1)$ in $\mathbb{Z}$, hence either q divides $k-1$ or q divides $k+1$ in $\mathbb{Z}$, so either $k = 1$ or $k = q - 1 = -1$ in $\mathbb{Z}_q$.

First of all, we have abelian groups of order 8*. By Fact 1 we get 3 of them:* $\mathbb{Z}_8$*,* $\mathbb{Z}_2 \times \mathbb{Z}_4$ *and* $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$*.*

Problem 4. *To deal with the nonabelian groups of order* 8 *we proceed as follows.*

(a) *Show that if* H *is a group such that every element of* H *different from* e *has order* 2*, then* H *is abelian.*

Solution: The assumption means that $a^2 = e$ for all $a \in H$, that is, $a = a^{-1}$. Then, for all $a, b \in H$, we have $ab = (ab)^{-1} = b^{-1}a^{-1} = ba$.

(b) *Show that if* H *is a group such that* $H/Z(H)$ *is cyclic, then* H *is abelian.*

Solution: Let $a \in H$ be a preimage of a generator of $H/Z(H)$ under the factor-map $H \to H/Z(H)$. Then every element of H can be written as $a^n b$ for some $n \in \mathbb{Z}$ and $b \in Z(H)$, and it is immediate that any two such elements commute.

Assume now that G *is a nonabelian group of order 8. By Fact 5 we know that its center is nontrivial, and it is not the entire group by assumption. By (b) above it cannot be of order 4 either (as then* $G/Z(G)$ *is of order 2, hence cyclic). Hence it is of order* 2*, so* $Z(G) = \{e, z\}$ *for some* z*.*

Next, since G *is not cyclic, it does not have elements of order* 8*. By (a) it cannot have only elements of order 1 and 2, so there must exist an element of order* 4*. Consider two cases.*

Case 1: *there is an element* $a \in G$ *of order* 4 *and an element* $b \in G \setminus \langle a \rangle$ *of order* 2*.*

(c) *Using arguments similar to those in Problem 3, conclude that* G *is isomorphic to a semidirect product* $\langle a \rangle \rtimes \langle b \rangle \cong \mathbb{Z}_4 \rtimes \mathbb{Z}_2$*, and then that* $G \cong D_8$*.*

Solution: Since $\langle a \rangle$ is a subgroup of index 2 in G, it is normal, and since $b \notin \langle a \rangle$, we have $G = \langle a \rangle \cup \langle a \rangle b$. Therefore the assumptions of 2b) are satisfied by the subgroups $\langle a \rangle \cong \mathbb{Z}_4$ and $\langle b \rangle \cong \mathbb{Z}_2$ of G and we get $G \cong \langle a \rangle \rtimes \langle b \rangle \cong \mathbb{Z}_4 \rtimes \mathbb{Z}_2$. The group $\mathbb{Z}_4$ has only two automorphisms (because it has only two generators 1 and $3 = -1$), the trivial one and the one mapping every element m into $-m$. Similarly to the solutions of 1c) and 3) we conclude that since G is nonabelian, we must have $G \cong D_8$.

Case 2: *for any element* a *of order* 4*, all elements in* $G \setminus \langle a \rangle$ *are of order* 4 *as well.*

Since our central element z *has order* 2*, we in particular must have* $z \in \langle a \rangle$*, and hence* $z = a^2$ *(since* a *and* a^3 *have order* 4*). We therefore get the following information about our group* G *of order* 8*: the center of* G *is generated by an element* z *of order* 2*, and every element* $a \in G \setminus Z(G)$ *has order* 4 *and satisfies* $a^2 = z$*.*

(d) *Show that such a group indeed exists, and any two such groups are isomorphic. Hint: for the existence, consider the subgroup of* $GL_2(\mathbb{C})$ *consisting of the matrices*

$$\pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \pm \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \pm \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

The group we thus obtain is called the ***quaternion group*** *and denoted by* Q_8 *or* Q*.*

Solution: Take an element a of order 4 and an element $b \in G \setminus \langle a \rangle$. As in the previous problem, $\langle a \rangle$ is a normal subgroup of G and we have $G = \langle a \rangle \cup \langle a \rangle b$, so

$$G = \{e, a, a^2 = z, a^3 = za, b, ab, a^2b = zb, a^3b = zab\} = \{z^i a^j b^k : i, j, k = 0, 1\}.$$

The automorphism $\operatorname{Ad} b$ cannot be trivial on $\langle a \rangle \cong \mathbb{Z}_4$, as then G would be abelian, hence, as we already used in Case 1, it must map a into $a^3 = za$. We thus have

$$a^2 = b^2 = z, \quad z^2 = e, \ ba = zab.$$

These relations are enough to write a product of two elements of the form $z^i a^j b^k$, $i, j, k = 0, 1$, in the same form, so these relations completely determine the multiplication table of G. It follows that there exists at most one group (up to isomorphism) with the above properties. According to the hint such a group indeed exists.

A variation of the above argument is as follows. Take a and b as above. Then $\langle a \rangle \cap \langle b \rangle = \{e, z\}$ and $G \setminus (\langle a \rangle \cup \langle b \rangle)$ consists of two elements. Put $c = ab$. Then $c \notin \langle a \rangle \cup \langle b \rangle$, as otherwise we would have $a \in \langle b \rangle$ or $b \in \langle a \rangle$ and then $\langle a \rangle = \langle b \rangle$, as both a and b are of order 4. We also clearly have $zc \notin \langle a \rangle \cup \langle b \rangle$ and $zc \neq c$. Hence

$$G = \{e, z, a, za, b, zb, c, zc\}.$$

We have the following relations:

$$a^2 = b^2 = c^2 = z, \quad z^2 = e, \quad ab = c.$$

One can check again that these relations are enough to completely reconstruct the multiplication table of G. For example, if we want to compute ba, then applying the operation of taking the inverse to both sides of $c = ab$ we get $zc = c^{-1} = b^{-1}a^{-1} = zbza = z^2ba = ba$. Hence G is unique up to isomorphism if it exists.

Finally, consider the groups of order 12. *Using again Fact 1 we get all such abelian groups:* $\mathbb{Z}_{12}$ *and* $\mathbb{Z}_2 \times \mathbb{Z}_6$.

Problem 5. *Assume* G *is a nonabelian group of order* 12.

(a) *Using Sylow theorems show that the number* N_2 *of Sylow* 2*-subgroups is* 1 *or* 3, *and the number* N_3 *of Sylow* 3*-subgroups is* 1 *or* 4. *Show then that the case* $N_2 = 3 \wedge N_3 = 4$ *is not possible. Hint: if* $N_3 = 4$, *then* G *has* 8 *elements of order* 3, *hence the remaining* 4 *elements form the unique Sylow* 2*-subgroup.*

Solution: The first statement follows from Sylow's third theorem. Now, assume $N_3 = 4$. Since a Sylow 3-subgroup of G is isomorphic to $\mathbb{Z}_3$, each such subgroup has two elements of order 3 and any two different such subgroups intersect trivially (that is, the intersection is $\{e\}$). It follows that we have at least 8 elements of order 3 in G. On the other hand, G must contain a Sylow 2-subgroup of order 4, which has no elements of order 3. Hence such a Sylow 2-subgroup must be the complement of those 8 elements of order 3, so $N_2 = 1$.

The case $N_2 = N_3 = 1$ *is not possible either, as then by the discussion in the class (from which we deduced Fact 4),* G *is isomorphic to the product of its Sylow subgroups, hence* G *is abelian.*

We have the following remaining possibilities. Choose Sylow 2- *and* 3*-subgroups* P_2 *and* P_3. *Note that* $|P_2| = 4$, *so* P_2 *is isomorphic to* $\mathbb{Z}_4$ *or* $\mathbb{Z}_2 \times \mathbb{Z}_2$, *and* $|P_3| = 3$, *so* P_3 *is isomorphic to* $\mathbb{Z}_3$.

Case 1: $N_2 = 1$ *and* $N_3 = 4$.

By Problem 2(b) we then have $G \cong P_2 \rtimes P_3$.

(b) *Show that* $\mathbb{Z}_4$ *has no automorphisms of order* 3, *so if we had* $P_2 \cong \mathbb{Z}_4$, *our group* G *would be abelian. Therefore the only possibility is* $P_2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. *Using Problem 1(b) show then that* $\mathbb{Z}_2 \times \mathbb{Z}_2$ *has exactly two automorphisms of order* 3, *and conclude that we get one group of order* 12 *in this case.*

As a bonus exercise show that this group is the alternating group A_4. *Hint: the unique Sylow* 2*-subgroup of* A_4 *consists of the identity and all products of two disjoint transpositions.*

Solution: We have already used in 4c) that $\mathbb{Z}_4$ has only two automorphisms.

By 1b) we have $\mathrm{Aut}(\mathbb{Z}_2 \times \mathbb{Z}_2) \cong S_3$. The group S_3 contains a unique subgroup of order 3, namely, A_3. It has two generators of order 3. Thus, $G \cong (\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes_\alpha \mathbb{Z}_3$, where α maps $1 \in \mathbb{Z}_3$ into one of the two elements of order 3 in $\mathrm{Aut}(\mathbb{Z}_2 \times \mathbb{Z}_2)$. The two semidirect product we thus get differ only by the choice of a generator of $\mathbb{Z}_3$, hence they are isomorphic and we get only one group (up to isomorphism) in this case. More formally, the two homomorphisms α_1 and α_2 that we get differ by the unique nontrivial automorphism β of $\mathbb{Z}_3$: $\alpha_1 = \alpha_2 \circ \beta$, - and then the map $(\mathrm{id}_{\mathbb{Z}_2 \times \mathbb{Z}_2}, \beta)$ of the set $(\mathbb{Z}_2 \times \mathbb{Z}_2) \times \mathbb{Z}_3$ onto itself defines an isomorphism

$$(\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes_{\alpha_1} \mathbb{Z}_3 \cong (\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes_{\alpha_2} \mathbb{Z}_3.$$

To see that the group we thus get is A_4, we follow the hint and observe that the identity and all products of two disjoint transpositions form a normal subgroup of A_4 (and even of S_4). This subgroup has 4 elements, so it is a Sylow 2-subgroup. By normality we conclude that $N_2 = 1$ and we are in the setting of Case 1.

Alternatively, by the solution of 1a) we know that Case 1 is characterized by the existence of 8 elements of order 3. In A_4 we do have such 8 elements - the cycles of length 3.

Case 2: $N_2 = 3$ *and* $N_3 = 1$.

By Problem 2(b) we then have $G \cong P_3 \rtimes P_2$.

We have the following subcases.

Subcase 2a: $P_2 \cong \mathbb{Z}_4$.

(c) *Show that* $\mathrm{Aut}(\mathbb{Z}_3) \cong \mathbb{Z}_2$. *Hence there exists a unique nontrivial homomorphism* $\mathbb{Z}_4 \to \mathrm{Aut}(\mathbb{Z}_3)$. *Conclude that in this case we get exactly one group* Dic_{12} *of order* 12. *This is an example of a* ***dicyclic group***.

Solution: The fact that $\mathbb{Z}_3$ has only two automorphisms is basically obvious (and has been used in 2c)), as $\mathbb{Z}_3$ has exactly two generators. The unique nontrivial homomorphism $\mathbb{Z}_4 \to \mathbb{Z}_2$ is obtained by mapping $1 \in \mathbb{Z}_4$ into $1 \in \mathbb{Z}_2$. Hence there is only one nonabelian semidirect product $\mathbb{Z}_3 \rtimes \mathbb{Z}_4$.

Subcase 2b: $P_2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

In this case the homomorphism $P_2 \to \mathrm{Aut}(P_3)$ defining the semidirect product becomes a nontrivial homomorphism $\mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathbb{Z}_2$.

(d) *Show that for every nontrivial homomorphism $f\colon \mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathbb{Z}_2$ there exists an automorphism g of $\mathbb{Z}_2 \times \mathbb{Z}_2$ such that $f \circ g$ has the form $(f \circ g)(x, y) = x$ for all $x, y \in \mathbb{Z}_2$. Conclude that the group G we get in this case is $D_6 \times \mathbb{Z}_2$.*
Show also that $D_6 \times \mathbb{Z}_2 \cong D_{12}$.

Solution: If $f\colon \mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathbb{Z}_2$ is a nontrivial homomorphism, then it is surjective. Take $a \in \mathbb{Z}_2 \times \mathbb{Z}_2$ such that $f(a) = 1$. Since the kernel of f has order 2, there also exists a nontrivial element b in the kernel. Then we define g by mapping $(1, 0)$ into a, $(0, 1)$ into b and $(1, 1)$ into $a + b$. This is indeed an automorphism of $\mathbb{Z}_2 \times \mathbb{Z}_2$ by 1b).

We thus see that any two nontrivial homomorphism $\alpha\colon \mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathrm{Aut}(\mathbb{Z}_3) \cong \mathbb{Z}_2$ differ by an automorphism of $\mathbb{Z}_2 \times \mathbb{Z}_2$, so similarly to 5b) we get only one semidirect product (up to isomorphism) in this case. Namely, by taking α such that $\alpha(0, 1) = \mathrm{id}$ and $\alpha(1, 0)$ is the unique nontrivial automorphism of $\mathbb{Z}_3$, we get $\mathbb{Z}_3 \rtimes_\alpha (\mathbb{Z}_2 \times \mathbb{Z}_2) = D_6 \times \mathbb{Z}_2$.

Finally, let a, b be the standard generators of D_6 and a', b' be the standard generators of D_{12}. Since a has order 3 and $1 \in \mathbb{Z}_2$ has order 2, the element $(a, 1) \in D_6 \times \mathbb{Z}_2$ has order 6. We also have $(b, 0)(a, 1)(b^{-1}, 0) = (a^{-1}, 1) = (a, 1)^{-1}$. Hence we can define a homomorphism $f\colon D_{12} \to D_6 \times \mathbb{Z}_2$ by

$$f(a') = (a, 1), \quad f(b') = (b, 0).$$

The image of f contains strictly more elements than $|\langle (a, 1) \rangle| = 6$, so its order divides 12 and is strictly larger than 6, hence the image must be the entire group $D_6 \times \mathbb{Z}_2$. Hence f is an isomorphism.

Alternatively, the inverse of f is given by $(a, 0) \mapsto (a')^{-2}$, $(b, 0) \mapsto b'$, $(0, 1) \mapsto (a')^3$.

We have thus proved that the following is a complete list of nontrivial groups G of order $n \leq 15$.

n	G
2	$\mathbb{Z}_2$
3	$\mathbb{Z}_3$
4	$\mathbb{Z}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_2$
5	$\mathbb{Z}_5$
6	$\mathbb{Z}_6$, $D_6 \cong S_3$
7	$\mathbb{Z}_7$
8	$\mathbb{Z}_8$, $\mathbb{Z}_4 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, D_8, Q_8
9	$\mathbb{Z}_9$, $\mathbb{Z}_3 \times \mathbb{Z}_3$
10	$\mathbb{Z}_{10}$, D_{10}
11	$\mathbb{Z}_{11}$
12	$\mathbb{Z}_{12}$, $\mathbb{Z}_6 \times \mathbb{Z}_2$, $D_{12} \cong D_6 \times \mathbb{Z}_2$, A_4, Dic_{12}
13	$\mathbb{Z}_{13}$
14	$\mathbb{Z}_{14}$, D_{14}
15	$\mathbb{Z}_{15}$