Section 15 - Factor groups

Let G be a group, $N \leq G$ a normal subgroup

Factor group $G/N = \{aN \mid a \in G\}$

$$aN\,bN = (ab)N \qquad \forall a, b \in G$$

Simple examples: • Let $N = G$. Then

$$G/N = G/G = \{aG \mid a \in G\} = \{G\},$$

the trivial group.

• Let $N = \{e\} \leq G$. Then

$$G/N = G/\{e\} = \{a\{e\} \mid a \in G\} = \{\{a\} \mid a \in G\}.$$

Multiplication is

$$a\{e\}\,b\{e\} = \{a\}\{b\} = (ab)\{e\} = \{ab\}.$$

The map $\gamma : G \to G/\{e\}$ given by $\gamma(a) = \{a\}$ is an isomorphism, $G \simeq G/\{e\}$.

Index 2 subgroups

Recall if $H \leq G$, index $(G:H) = |G/H| = \#$ of left cosets.

Thm Let G be a finite group, $H \leq G$ s.t. $(G:H) = 2$.

Then H is normal.

Proof: H has 2 left cosets, each of size $|H|$, so

$$|G| = 2|H|.$$

One coset is H, the other must be $G \setminus H$,

Similarly $|G| = 2|H|$ means H has 2 right cosets H & $G \setminus H$.

Let $a \in G$. If $a \in H$, then $aH = Ha = H$

If $a \notin H$, then $eH = G \setminus H = Ha$.

Hence H is normal.

Example: $A_n \trianglelefteq S_n$ has $|S_n| = n! = 2|A_n|$, so A_n is normal in S_n

$$|S_n/A_n| = 2 \implies S_n/A_n \simeq \mathbb{Z}_2.$$

Example: There is no $H < A_4$ with $|H| = 6$.

Proof: $|A_4| = \frac{|S_4|}{2} = \frac{4!}{2} = 12$. $|H| = 6 \implies H$ is normal, $A_4/H \simeq \mathbb{Z}_2$.

In $\mathbb{Z}_2$, have $x + x = 0$ for each $x \in \mathbb{Z}_2$.

$\Downarrow$

In A_4/H, $x^2 = e$ for each $x \in A_4/H$.

$\Downarrow$

Let $a \in A_4$. Then $aHaH = eH$.

By def. $aHaH = a^2H$, so must have $a^2 \in H$.

So: H contains a^2 for each $a \in A_4$.

$(1\,2\,3)^2 = (1\,3\,2)$ $(1\,2\,4)^2 = (1\,4\,2)$ $(1\,3\,4)^2 = (1\,4\,3)$

$(1\,3\,2)^2 = (1\,2\,3)$ $(1\,4\,2)^2 = (1\,2\,4)$ $(1\,4\,3)^2 = (1\,3\,4)$

$(2\,3\,4)^2 = (2\,4\,3)$ $(2\,4\,3)^2 = (2\,3\,4)$

H contains all 8 elements above, contradicting $|H| = 6$.

Thm Let H, K be groups, $H \times K = \{(h,k) \mid h \in H, k \in K\}$ their direct product. Let $\bar{H} = \{(h, e_K) \mid h \in H\}$. Then

1. $\bar{H} \simeq H$
2. $\bar{H}$ is a normal subgroup
3. $H \times K / \bar{H} \simeq K$

Proof ① $\phi : H \to \bar{H}$ given by $\phi(h) = (h, e_K)$ is an isomorphism

②+③: Let $\pi_2 : H \times K \to K$ be given by

$$\pi_2(h,k) = k.$$

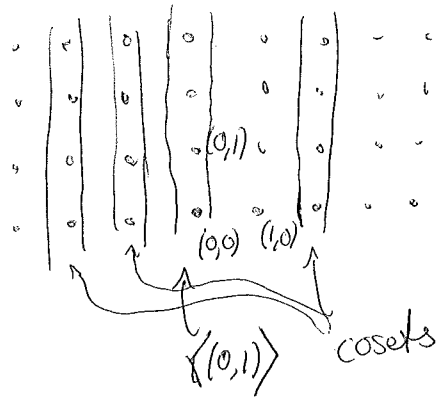
Then have diagram

$$\begin{array}{ccc} H \times K & \xrightarrow{\pi_2} & \pi_2[H \times K] \leq K \\ & \searrow^{\gamma} \quad \nearrow_{\mu} & \\ & H \times K / \ker \pi_2 & \end{array}$$

Here $\pi_2[H \times K] = K$ (π_2 is onto), $\ker \pi_2 = \bar{H}$, so $\bar{H}$ is normal and $H \times K / \bar{H} \simeq K$.

(Explicitly, elements of $H \times K / \bar{H}$ are sets of the form $(e_H, k)\bar{H} = \{(h,k) \mid h \in H\}$, $(e_H, k)\bar{H}(e_H, k')\bar{H} = (e_H, kk')\bar{H}$.)

Example $\mathbb{Z} \times \mathbb{Z}$, $\langle (0,1) \rangle < \mathbb{Z} \times \mathbb{Z}$. $\langle (0,1) \rangle$ is normal since $\mathbb{Z} \times \mathbb{Z}$ is abelian. $\mathbb{Z} \times \mathbb{Z} / \langle (0,1) \rangle \simeq \mathbb{Z}$



Finitely generated abelian groups

Thm Let G be a cyclic group, $N \leq G$. Then N is normal & G/N is cyclic.

Pf: G cyclic $\Rightarrow$ G abelian $\Rightarrow$ N normal.

Let $a \in G$ be s.t. $G = \langle a \rangle$.

Elements of G/N are bN, $b \in G$.

Write $b = a^n$, $n \in \mathbb{Z}$, then

$$bN = a^n N = (aN)^n,$$

hence Every element of G/N is of form $(aN)^n \iff \langle aN \rangle = G/N$.

Thm Let G be finitely generated & abelian, $N \leq G$. Then G/N is finitely generated and abelian.

Pf: G abelian $\Rightarrow ab = ba \quad \forall a, b \in G$

$\Rightarrow aN\,bN = (ab)N = (ba)N = bN\,aN, \quad \forall a, b \in G \Rightarrow G/N$ abelian.

G finitely generated means there is $g_1, \dots, g_r \in G$ s.t. any $g \in G$ can be written as

$$g = g_{i_1}^{e_1} g_{i_2}^{e_2} \cdots g_{i_\ell}^{e_\ell} \qquad i_j \in \{1, \dots, r\}, \quad e_j \in \mathbb{Z}.$$

Then

$$gN = (g_{i_1}^{e_1} \cdots g_{i_\ell}^{e_\ell})N = (g_{i_1}N)^{e_1} \cdots (g_{i_\ell}N)^{e_\ell}$$

$\Rightarrow g_1 N, \dots, g_r N$ generate G/N.

General question: Given G finitely generated abelian group, $\exists\, N \leq G$, ∇

which fin.gen. ab. gp. is G/N?

Recall a fin. gen. ab. gp. is isomorphic to a unique one of the form

$$\mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z} \times \mathbb{Z}_{p_1^{e_1}} \times \cdots \times \mathbb{Z}_{p_k^{e_k}} \qquad p_i \text{ primes}, \; e_i \in \mathbb{Z}^+.$$

Example $\langle (2,1) \rangle < \mathbb{Z}_4 \times \mathbb{Z}_2$, what group is $\mathbb{Z}_4 \times \mathbb{Z}_2 / \langle (2,1) \rangle$?

Note order of $(2,1)$ is 2, since $2(2,1) = (2 +_4 2, 1 +_2 1) = (0,0)$.

Hence $\langle (2,1) \rangle = \{(0,0), (2,1)\}$, $|\langle (2,1) \rangle| = 2$, so

$$\left| \mathbb{Z}_4 \times \mathbb{Z}_2 / \langle (2,1) \rangle \right| = \frac{|\mathbb{Z}_4 \times \mathbb{Z}_2|}{|\langle (2,1) \rangle|} = \frac{8}{2} = 4.$$

Up to isomorphism, have 2 groups of order 4; $\mathbb{Z}_4$ & $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Which one is $\mathbb{Z}_4 \times \mathbb{Z}_2 / \langle (2,1) \rangle$ isomorphic to?

Note in $\mathbb{Z}_2 \times \mathbb{Z}_2$, have $x + x = 0$ for all x, while

in $\mathbb{Z}_4$, $1 + 1 = 2 \neq 0$.

In $\mathbb{Z}_4 \times \mathbb{Z}_2 / \langle (2,1) \rangle$ have an element $(1,0) + \langle (2,1) \rangle$, such that

$$\big((1,0) + \langle (2,1) \rangle\big) + \big((1,0) + \langle (2,1) \rangle\big) = (2,0) + \langle (2,1) \rangle.$$

Since $(2,0) \notin \langle (2,1) \rangle$, the coset $(2,0) + \langle (2,1) \rangle$ is not equal to $\langle (2,1) \rangle$.

Hence $(1,0) + \langle (2,1) \rangle + (1,0) + \langle (2,1) \rangle \neq \langle (2,1) \rangle = 0$, so $\mathbb{Z}_4 \times \mathbb{Z}_2 / \langle (2,1) \rangle \neq \mathbb{Z}_2 \times \mathbb{Z}_2$.

Then must have $\mathbb{Z}_4 \times \mathbb{Z}_2 / \langle (2,1) \rangle \cong \mathbb{Z}_4$.

Simple groups

Def A group G is simple if it contains only $\{e\}$ & G as normal subgroups, and $G \neq \{e\}$.

Examples: If G is abelian, then G is simple if and only if $G \cong \mathbb{Z}_p$.

Proof Assume G is abelian & simple. Let $a \in G$, $a \neq e$. Then $\langle a \rangle$ is a subgroup of G, which is normal since G is abelian. Since G is simple and $\langle a \rangle \neq \{e\}$, must have $\langle a \rangle = G$. So G is cyclic, hence either $G \cong \mathbb{Z}$ or $G \cong \mathbb{Z}_n$ for some n. $\mathbb{Z}$ is not simple, since $2\mathbb{Z} < \mathbb{Z}$ is a normal subgroup.

~~$\mathbb{Z}_n$ is not simple if n is not prime since if $n = p_1 p_2$ with p_1, p_2 distinct primes, have $\mathbb{Z}_{p_1}$~~

If n is not prime, then $\mathbb{Z}_n$ is not simple: Let $1 < p < n$ be a prime dividing n, then $\langle p \rangle = \{p, 2p, \dots, n-p\} < \mathbb{Z}_n$ is a proper subgroup. Hence we must have $G \cong \mathbb{Z}_p$ for some prime p.

To see that $\mathbb{Z}_p$ is simple, use Lagrange's theorem: A subgroup $N < \mathbb{Z}_p$ must have order dividing $|\mathbb{Z}_p| = p$, which means $|N| = p$ or $|N| = 1$. In the first case, $N = \mathbb{Z}_p$, in the second $N = \{0\}$.

Thm The group A_n is simple if $n \geq 5$

Proof: See exercise in Fraleigh.

Note S_n is "never" simple, since $A_n < S_n$ is normal.

* S_2 is simple

Well,

For "culture":

Feit-Thompson Theorem: A nonabelian finite simple group has even order.

Classification of finite simple groups: A finite simple group G is isomorphic to one of

1. $\mathbb{Z}_p$, p prime
2. A_n, $n \geq 5$
3. A group of "Lie type" } explicit subclassification of these
4. 26 "sporadic groups".

Final sporadic group, the "monster group" M, constructed in 1980,

$$|M| \approx 8 \cdot 10^{53}$$

Classification considered complete ca. 2004.