Section 16 - Group actions

So, why study groups?

"Main" reason: Groups "act" on other things.

Def Let G be a group, X a set. An **action of G on X** is a map $*: G \times X \longrightarrow X$, usually written

$$*(g,x) = gx \qquad g \in G,\ x \in X$$

satisfying

① $ex = x \qquad \forall x \in X$

② $g_1(g_2 x) = (g_1 g_2)x \qquad \forall g_1, g_2 \in G,\ x \in X.$

A set X with a chosen action $*$ of G is called a **G-set**.

Example Let X be a set. S_X acts on X, via

$$*(\sigma, x) = \sigma x = \sigma(x) \qquad \sigma \in S_X,\ x \in X.$$

Check ① Identity $\iota \in S_X$ has $\iota(x) = x \quad \forall x \in X.$

$$\iota x = \iota(x) = x, \qquad \text{o.k.}$$

② Given $\sigma_1, \sigma_2 \in S_X$, have $x \in X$, have

$$\sigma_1(\sigma_2 x) = \sigma_1(\sigma_2(x)).$$

$$(\sigma_1 \sigma_2)(x) = (\sigma_1 \circ \sigma_2)(x) = \sigma_1(\sigma_2(x)),$$

so $\sigma_1(\sigma_2(x)) = (\sigma_1 \sigma_2)(x)$. o.k.

Example $GL(n, \mathbb{R}) = \{\text{invertible real } (n \times n)\text{-matrices}\}$ acts on $\mathbb{R}^n$ via

$$*(A, \vec{v}) = A\vec{v} \qquad A \in GL(n,\mathbb{R}),\ \vec{v} \in \mathbb{R}^n$$

Check: (1) Identity in $GL(n,\mathbb{R})$ is $I_n = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$,

$I_n \vec{v} = \vec{v} \quad \forall \vec{v} \in \mathbb{R}^n$, o.k.

(2) $A(B\vec{v}) = (AB)\vec{v}$, o.k.

Actions on X are homomorphisms to S_X

Thm Let X be a G-set, $g \in G$. The map

$$\sigma_g : X \longrightarrow X$$

$$\sigma_g(x) = gx$$

is a bijection, (so $\sigma_g \in S_X$).

Proof ~~Consider σ_e. Have $\sigma_e(x) = ex = x \quad \forall x \in X$, so σ_e is bijective, and equal to identity $\in S_X$.~~

For σ_g, $x \in X$

$$\sigma_g(\sigma_{g^{-1}}(x)) = \sigma_g(g^{-1}x) = g(g^{-1}x) = (gg^{-1})x = ex = x$$

$$\sigma_{g^{-1}}(\sigma_g(x)) = \sigma_{g^{-1}}(gx) = g^{-1}(gx) = (g^{-1}g)x = ex = x.$$

Hence $\sigma_{g^{-1}}$ is a 2-sided inverse to σ_g, so σ_g is bijective. □

Thm Let X be a G-set. The map

$$\phi : G \longrightarrow S_X$$

$$\phi(g) = \sigma_g$$

is a homomorphism.

Proof: $\phi(g_1 g_2) = \sigma_{g_1 g_2} \overset{*}{=} \sigma_{g_1} \circ \sigma_{g_2} = \phi(g_1)\,\phi(g_2)$.

$*$ since $\sigma_{g_1 g_2}(x) = (g_1 g_2)x = g_1(g_2 x) = \sigma_{g_1}(\sigma_{g_2}(x))$. □

Note Conversely, any homomorphism $\phi : G \longrightarrow S_X$ gives an action of G on X, via

$$gx = \phi(g)(x).$$

Given group G & set X

$$\{\text{Actions of } G \text{ on } X\} \xleftrightarrow{\text{bijection}} \{\text{Homomorphisms } \phi: G \to S_X\}.$$

Faithful & transitive actions

Thm Let X be a G-set. Then

$$\{g \in G \mid gx = x \text{ for all } x \in X\} \subseteq G$$

is a normal subgroup of G.

Proof $gx = x$ for all $x \in X$

$\Updownarrow$

$\sigma_g(x) = x \quad \forall x \in X$

$\Updownarrow$

$\sigma_g = \iota \in S_X$

$\Updownarrow$

$\phi(g) = \iota$

$\Updownarrow$

$g \in \ker \phi$.

The set equals $\ker \phi$, a normal subgroup.

Def If X is a G-set and

$$\{g \in G \mid gx = x \quad \forall x \in X\} = \{e\},$$

say G acts faithfully on X.

Def If X is a G-set & given any $x_1, x_2 \in X$, there is a $g \in G$ such that

$$gx_1 = x_2,$$

say G acts transitively on X.

Example Let G be a group. Then G acts on G via multiplication:

$$gx = g \overset{\curvearrowleft}{x}$$

(gx: G-action; $g \overset{\curvearrowleft}{x}$: group product.)

This action is faithful:

$gx = x \quad \forall x \in X \implies g = e$

Action is transitive: Given $x_1, x_2 \in G$, have $x_2 x_1^{-1} \in G$ s.t.

$$(x_2 x_1^{-1}) x_1 = x_2.$$

Action faithful $\iff$ $\phi : G \to S_G$ has $\ker \phi = \{e\}$

$\iff$ ϕ is one-to-one

$\iff$ $G \cong \phi[G] \leq S_G$ (Cayley's theorem!)

Isotropy groups, fix point sets & orbits

Let G be a group, X a G-set.

For $g \in G$, $X_g = \{x \in X \mid gx = x\}$, the "fix point set of g".

For $x \in X$, $G_x = \{g \in G \mid gx = x\}$, the "isotropy subgroup of x."

G_x is a subgroup of G: $ex = x$ (identity ✓)

$g_1 x = x$, $g_2 x = x \Rightarrow (g_1 g_2)x = g_1(g_2 x) = g_1 x = x$ (closed ✓)

$gx = x \Rightarrow g^{-1}gx = g^{-1}x \Rightarrow x = g^{-1}x$ (inverses ✓).

If $x \in X$, define the orbit of x as

$$Gx = \{gx \mid g \in G\} \subseteq X.$$

If $y \in Gx$, then there is a $g \in G$ s.t. $y = gx$.

Hence $g^{-1}y = g^{-1}gx \Rightarrow g^{-1}y = x \Rightarrow x \in Gy$.

X is partitioned into its orbits.

Example: $X = \{1, 2, \ldots, 6\}$, $G = \mathbb{Z}_6$. Action defined by

homomorphism $\phi : \mathbb{Z}_6 \to S_6$ $\quad \phi(i) = \sigma^i$, where $\sigma = (123)(45)$.

Multiplication table S_6 ↷

$\mathbb{Z}_6 \Rightarrow$

	1	2	3	4	5	6
0	1	2	3	4	5	6
1	2	3	1	5	4	6
2	3	1	2	4	5	6
3	1	2	3	5	4	6
4	2	3	1	4	5	6
5	3	1	2	5	4	6
6						

Orbits are

$\mathbb{Z}_6 1 = \mathbb{Z}_6 2 = \mathbb{Z}_6 3 = \{1, 2, 3\}$

$\mathbb{Z}_6 4 = \mathbb{Z}_6 5 = \{4, 5\}$

$\mathbb{Z}_6 6 = \{6\}$.

Relation between orbits & isotropy group: Roughly

G_x small $\Longleftrightarrow$ Gx big.

Precise version, thm: Let X be a G-set, $x \in X$. There is a bijection

$$G/G_x \longleftrightarrow Gx,$$

defined* by

$$aG_x \longleftrightarrow ax$$

$\forall$ a eg.

If G & X are finite,

$$|G| = |Gx|\,|G_x|.$$

* It is not obvious this is well-defined!