Section 17 - Counting via group actions

Recall action of group G on set X is map

$$*: G \times X \longrightarrow X, \quad \text{written } *(g,x) = gx,$$

such that

① $ex = x$ $\quad \forall x \in X$

② $g_1(g_2 x) = (g_1 g_2)x$ $\quad \forall g_1, g_2 \in G, \ x \in X.$

For $g \in G$, $X_g = \{x \in X \mid gx = x\} \subseteq X$, fix points of g

For $x \in X$, $G_x = \{g \in G \mid gx = x\} \leq G$, isotropy group of x.

A G-set X is partitioned into <u>orbits</u>, where the orbit containing $x \in X$ is

$$Gx = \{gx \mid g \in G\}.$$

<u>Thm</u> Let X be a G-set, $x \in X$. Then have bijection

$$Gx \longleftrightarrow \{\text{left cosets of } G_x\} = G/G_x.$$

<u>Proof</u> Map $\phi: Gx \longrightarrow G/G_x$ by

$$\phi(gx) = gG_x.$$

Well-defined? If $gx = g'x$, then $(g')^{-1}gx = x \Rightarrow (g')^{-1}g \in G_x \Rightarrow g'G_x = gG_x$. ✓

Map $\psi: G/G_x \longrightarrow Gx$ by

$$\psi(gG_x) = gx.$$

Well-defined? If $gG_x = g'G_x$, then $(g')^{-1}g \in G_x, \Rightarrow (g')^{-1}gx = x$

$$\Rightarrow gx = g'x.$$

Clearly ϕ & ψ are inverse to one another, so define a bijection.

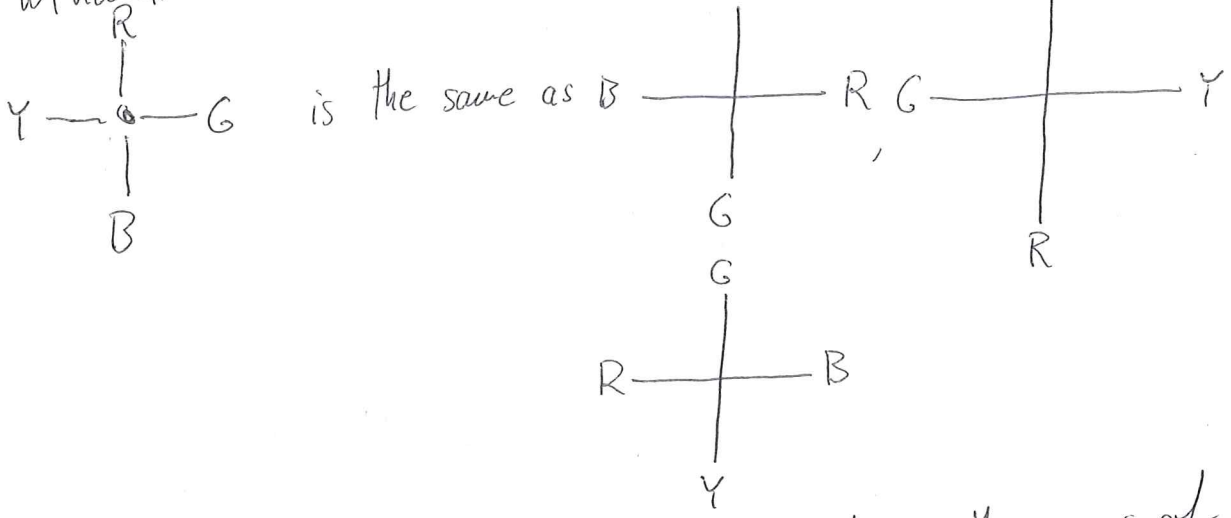
Example: If $G_x = \{e\}$, then Gx is in bijection with $G/\{e\} \simeq G$.

- If $G_x = G$, then Gx is in bijection with $G/G = \{G\}$, so $Gx = \{x\}$.

Corollary If G is finite, then $|Gx| = \frac{|G|}{|G_x|}$.

Proof $|G/G_x| = \frac{|G|}{|G_x|}$.

Example A counting problem: A windmill has 4 arms. You want to paint each arm a separate colour, using the colours red, green, blue & yellow. You consider two colourings the same if they are related by rotating the arms of the windmill:

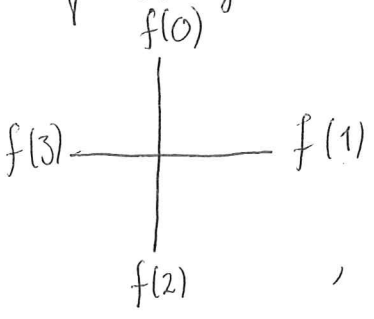


In how many distinct ways can you colour the arms of the windmill?

Let X be the set of bijective functions

$$f : \{0, 1, 2, 3\} \longrightarrow \{R, G, B, Y\}$$

Elements of X give colourings of the windmill by



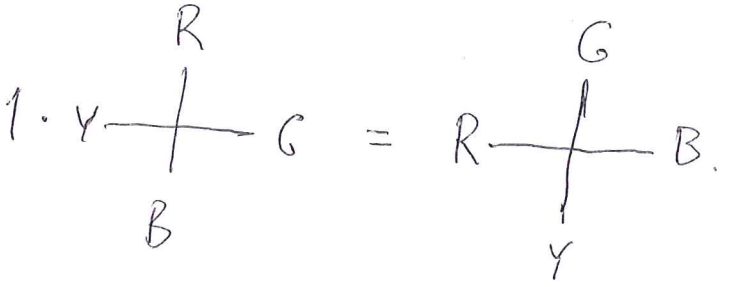
but different f can give the same colouring, for example, if $g(0) = f(1)$, $g(1) = f(2)$, $g(2) = f(3)$ & $g(3) = f(0)$, then f & g define the same colouring.

There is an action of $\mathbb{Z}_4$ on X by

$$(i\,f)(x) = f(x +_4 i).$$

$i \in \mathbb{Z}_4$
$x \in \{0, 1, 2, 3\}$
$f \in X$

In pictures, multiplying by $i \in \mathbb{Z}_4$ rotates the colouring defined by f i steps counter-clockwise:



This means that given $f, g \in X$, they define the same colouring if and only if there is an $i \in \mathbb{Z}_4$ such that $if = g$, which is if and only if f & g lie in the same $\mathbb{Z}_4$-orbit. So the number of colourings equals the number of $\mathbb{Z}_4$-orbits in X.

Burnside's formula: Let G be a finite group, X a finite G-set. Let r be the number of G-orbits. Then

$$r = \frac{\sum_{g \in G} |X_g|}{|G|}.$$

Note: Remember this as $r =$ "average no. of fix points for an element in G."

Proof Consider

$$S = \{(g, x) \in G \times X \mid gx = x\}$$

Have $S = \{(g,x) \in G \times X \mid x \in X_g\}$, so $|S| = \sum_{g \in G} |X_g|$.

Have $S = \{(g,x) \in G \times X \mid g \in G_x\}$, so $|S| = \sum_{x \in X} |G_x|$.

RHS of formula is

$$\frac{\sum_{g \in G} |X_g|}{|G|} = \frac{\sum_{x \in X} |G_x|}{|G|} = \sum_{x \in X} \frac{|G_x|}{|G|} = \sum_{x \in X} \frac{1}{(G : G_x)} = \sum_{x \in X} \frac{1}{|Gx|}.$$

Let the orbits of X be $Gx_1, Gx_2, \ldots, Gx_r$, then

$$\sum_{x \in X} \frac{1}{|Gx|} = \sum_{x \in Gx_1} \frac{1}{|Gx|} + \cdots + \sum_{x \in Gx_r} \frac{1}{|Gx|}$$

$$\overset{*}{=} \sum_{x \in Gx_1} \frac{1}{|Gx_1|} + \cdots + \sum_{x \in Gx_r} \frac{1}{|Gx_r|} = 1 + \cdots + 1 = r,$$

* using $x \in Gx_i \Rightarrow Gx = Gx_i$.

q.e.d.

In our example, $G = \mathbb{Z}_4$, $X = \{\text{bijections } f : \{0,1,2,3\} \to \{R,G,B,Y\}\}$

$|X| = 4! = 24.$

Fix point sets:

$$X_0 = \{ f \in X \mid 0f = f \} = X, \quad |X_0| = 24$$

$$X_1 = \{ f \in X \mid 1f = f \} = \{ f \in X \mid f(x+1) = f(x) \text{ for all } x \}$$

Since elements of X are bijections, we get

$$X_1 = \emptyset, \quad |X_1| = 0.$$

Similarly $|X_2| = |X_3| = 0.$

The number of orbits is

$$\frac{|X_0| + |X_1| + |X_2| + |X_3|}{|\mathbb{Z}_4|} = \frac{24}{4} = \underline{\underline{6}}.$$