Forelesning, 12. februar 2024

Def 13.1

$\varphi: G \to G'$ homomorfi $\iff \varphi(ab) = \varphi(a)\varphi(b) \quad \forall a,b \in G$

Ex $\varphi: G \to G'$ surjektiv, G abelsk $\Rightarrow$ G' er abelsk

Bevis: $a', b' \in G' \Rightarrow \exists a, b \in G$ slik at $a' = \varphi(a)$, $b' = \varphi(b)$

$ab = ba \Rightarrow \varphi(ab) = \varphi(ba)$

$\Rightarrow a'b' = b'a'$

Ex $F = \{f: \mathbb{R} \to \mathbb{R}\}$ additive gruppe

$\mathbb{R}$: add. gr av reelle tall

$\varphi_c: F \to \mathbb{R}$ gitt ved $\varphi_c(f) = f(c)$

$\varphi_c(f+g) = f(c) + g(c) = \varphi_c(f) + \varphi_c(g)$

Ex $T: \mathbb{R}^n \to \mathbb{R}^m$ gitt ved $T(x) = Ax$

T homomorfi: $T(x+y) = A(x+y) = Ax + Ay = T(x) + T(y)$

Ex $\det: GL_n(\mathbb{R}) \to \mathbb{R}^*$

$\det(AB) = \det(A)\det(B)$

homomorfi.

Ex $\varphi_r: \mathbb{Z} \to \mathbb{Z}$, $1 \mapsto n$

homomorfi

$\varphi_r(a+b) = r(a+b) = ra + rb = \varphi_r(a) + \varphi_r(b)$

Ex $\pi_i: G_1 \times \cdots \times G_n \to G_i$

projeksjon

Ex $\sigma: F \to \mathbb{R}$, $F = \{f: [0,1] \to \mathbb{R}\}$

Evaluering: $\sigma(f) = \int_0^1 f\,dx$

$\sigma(f+g) = \int_0^1 (f+g)\,dx = \int_0^1 f\,dx + \int_0^1 g\,dx$

$= \sigma(f) + \sigma(g)$.

Ex $\gamma: \mathbb{Z} \to \mathbb{Z}_m$ $\quad \gamma(n) = n \pmod m$

$1 \longmapsto \bar{1}$

Def 13.11 $\varphi: X \to Y$, $A \subseteq X$

$\varphi[A]$ billede av A under φ

$\varphi[X]$ (range)

$\varphi^{-1}(B)$ invers billede

Thm 13.12 $\varphi: G \to G'$ gruppehomomorf

1) $\varphi(e) = e'$ identitetselement
2) $\varphi(g^{-1}) = \varphi(g)^{-1}$ invers
3) $H \leq G \Rightarrow \varphi[H] \leq \varphi[G] \leq G'$
4) $K' \leq G' \Rightarrow \varphi^{-1}(K') \leq G$

Bevis for 4).

Lukket. i) $a, b \in \varphi^{-1}(K') \Rightarrow \varphi(a), \varphi(b) \in K'$

$\Rightarrow \varphi(ab) \in K' \Rightarrow ab \in \varphi^{-1}(K')$

identitet ii) $e \in G \Rightarrow \varphi(e) = e' \in K' \Rightarrow e \in \varphi^{-1}(K')$

invers iii) $g \in \varphi^{-1}(K') \Rightarrow \varphi(g) \in K' \Rightarrow \varphi(g)^{-1} \in K'$

$\Rightarrow g^{-1} \in \varphi^{-1}(K')$ $\quad \varphi(g^{-1}) \in K'$

Def 13.13 $\varphi: G \to G'$ gruppehomomorfi

$\Rightarrow \varphi^{-1}[e'] = \ker \varphi$ Kjernen gruppe

$\{e'\} \leq G'$ trivielle undergruppe

Thm 13.15 $\varphi: G \to G'$ gruppehomomorfi, $H = \ker(\varphi) \leq G$

$\Rightarrow \varphi^{-1}(\varphi(a)) = \begin{cases} aH \\ Ha \end{cases}$

Bevis $x \in \varphi^{-1}(\varphi(a)) \Rightarrow \varphi(x) = \varphi(a) \Rightarrow \varphi(xa^{-1}) = \varphi(e) = e'$

$\Rightarrow xa^{-1} \in \ker(\varphi) = H \Rightarrow x \in Ha$

Alt. $a^{-1}x \in H \Rightarrow x \in aH$

Ex $F=\{f: \mathbb{R}\to\mathbb{R}\}$ $\quad \varphi: D \to F$

$D=\{f\in F \mid f \text{ deriverbar}\}$ $\quad$ derivasjon gruppehomomorfi

$H = \ker(\varphi) = \{\text{konstante funksjoner}\}$

$\varphi^{-1}(\varphi(f)) = f + C$ dvs. restklasse $f+H$

Kor 13.18 $\quad \varphi: G\to G'$ gruppehomomorfi $\quad \varphi$ 1-1 $\iff \ker(\varphi)=\{e\}$

Bevis φ 1-1 betyr $a\neq b$ gir $\varphi(a)\neq\varphi(b)$

Vi har $\varphi(e)=e'$ identitet, og derfor hvis $\varphi(a)=e'$ så er $a=e$

$\ker(\varphi)=\{e\}$ betyr: $a\neq b\in G \Rightarrow ab^{-1}\neq e$

$\Rightarrow \varphi(ab^{-1})\neq e' \Rightarrow \varphi(a)\varphi(b)^{-1}\neq e'$

$\Rightarrow \varphi(a)\neq\varphi(b)$

QED

Def 13.19 $\quad H\leq G \iff \forall g\in G,\ gH=Hg$. $\quad$ normal

Kor 13.20 $\quad \ker(\varphi)\leq G$ er normal

Def 14.15 En gruppeisomorfi $\varphi: G\to G$ kalles en AUTOMORFI

Avbildningen: La $g\in G$; $\varphi: G\to G$ gitt ved

$\varphi(h) = g^{-1}hg = i_g(h)$ Konjugasjon med g.

kalles en INDRE AUTOMORFI

i_g er en isomorfi 1) $i_g(xy) = g^{-1}xyg = g^{-1}xgg^{-1}yg = i_g(x)\,i_g(y)$

2) $i_g(x)=i_g(y) \Rightarrow g^{-1}xg = g^{-1}yg \Rightarrow x=y$ injektiv

3) $\forall g\in G$; $i_g(gyg^{-1}) = g^{-1}gyg^{-1}g = y$ surjektiv

$H \leq G \qquad i_g(H) = g^{-1}Hg \leq G$. Konjugerat undergrupp av H

G abelsk $\Rightarrow \quad i_g(H) = g^{-1}Hg = g^{-1}gH = H \quad \forall g \in G$

$H \leq G$ normal $\Rightarrow \quad i_g(H) = g^{-1}Hg = H \quad \forall g \in G$

$|G| < \infty \Rightarrow \quad |g^{-1}Hg| = |H|$

Ty $i_g : H \longrightarrow g^{-1}Hg$

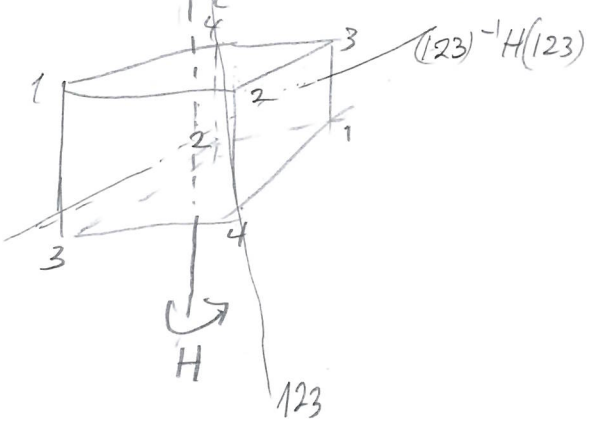
- $i_g(xy) = g^{-1}xyg = g^{-1}xgg^{-1}yg = i_g(x)\,i_g(y)$
- $i_g(e) = g^{-1}eg = g^{-1}g = e$
- $i_g^{-1}(g^{-1}hg) = h$.

$\forall g \in G;\ H \simeq g^{-1}Hg$

Ex $\quad S_4 \geq H = \{id, (1234), (13)(24), (1432)\} \simeq \mathbb{Z}_4$

$(123)^{-1}H(123) = \{id, (132)(1234)(123), (132)(13)(24)(123), (132)(1432)(123)\}$

$= \{id, (1243), (14)(23), (1342)\} \simeq \mathbb{Z}_4$





$S_3 \geq H = \{id, (12)\}$

$(123)^{-1}H(123) = \{id, (132)(12)(123)\} = \{id, (13)\}$

