Forelesning 19. februar

$$GL_n(\mathbb{R}) = \{ M \in M_{n\times n}(\mathbb{R}) \mid \det(M) \neq 0 \}$$

Gruppe fordi $\det(M \cdot N) = \det(M)\det(N)$

Undergrupper:

$$SL_n(\mathbb{R}) = \{ M \in GL_n(\mathbb{R}) \mid \det(M) = 1 \}$$

$$O_n(\mathbb{R}) = \{ M \in GL_n(\mathbb{R}) \mid M^{-1} = M^t \}$$

$$SO_n(\mathbb{R}) = \{ M \in O_n(\mathbb{R}) \mid \det(M) = 1 \}$$

Ortogonale gruppa $O_2(\mathbb{R})$:

$v \in \mathbb{R}^2$: $\|v\|^2 = v^T v$

Basisskifte $w = Av$ slik at

$\|v\| = \|w\|$

$$\Rightarrow \quad w^T w = (Av)^T (Av) = v^T A^T A v = v^T \cdot v \quad \forall v$$

$$v^T \begin{pmatrix} a & b \\ c & d \end{pmatrix} v = v^T \cdot v \qquad v = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$(x\ y)\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = ax^2 + (b+c)xy + dy^2 = x^2 + y^2 \quad \forall x, y \in \mathbb{R}$$

$$\Rightarrow \quad (a-1)x^2 + (b+c)xy + (d-1)y^2 = 0 \quad \forall x, y \in \mathbb{R}$$

Sett $y = 0 \Rightarrow (a-1)x^2 = 0$ dvs $a = 1$

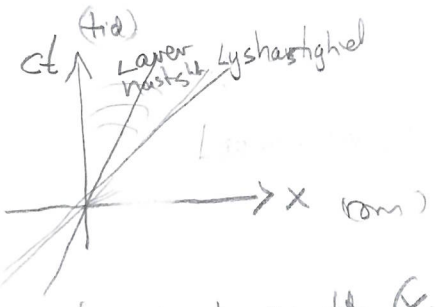
$x = 0 \Rightarrow (d-1)y^2 = 0$ dvs $d = 1$

$$\Rightarrow \quad (b+c)xy = 0 \quad \text{dvs } b + c = 0$$

$A^T \cdot A = \begin{pmatrix} 1 & b \\ -b & 1 \end{pmatrix}$ Men $(A^T A)^T = A^T A$ som betyr $b = 0$

$\Rightarrow A^T A = id$. ella $\underline{\underline{A^t = A^{-1}}}$

Tidrommet



Koordinater i et punkt (x, ct) $\quad (ct)^2 - x^2$

Basisskifte i nye koordinat (x', ct') $\quad (ct')^2 - (x')^2$ $\Rightarrow$ Lyshastighet samme i alle referansesystem

$$(Av)^T \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} Av = v^T \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} v$$

ds $\quad A^T \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ gir $\begin{pmatrix} a & c \\ b & d \end{pmatrix} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$\begin{pmatrix} a & -c \\ b & -d \end{pmatrix}$

$a^2 - c^2 = 1 \quad ab - cd = 0$

$ab - cd = 0 \quad b^2 - d^2 = -1$

Lorentzgruppa $\Rightarrow \quad A = \begin{pmatrix} a & b \\ b & a \end{pmatrix} \quad a^2 - b^2 = 1 \quad$ (f.eks $a = \cosh u$, $b = \sinh u$)

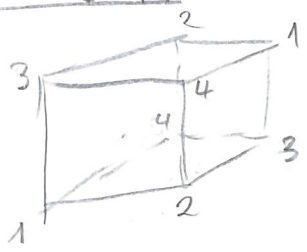
$A \begin{pmatrix} ct \\ vt \end{pmatrix} = \begin{pmatrix} ? \\ 0 \end{pmatrix} \Rightarrow \quad t(bc + av) = 0 \Rightarrow b = -\frac{av}{c}$

$a^2 - b^2 = 1$ gir $a^2\left(1 - \frac{v^2}{c^2}\right) = 1$

$(ac + bv)t = \left(ac - \frac{av^2}{c}\right)t = ac\left(1 - \frac{v^2}{c^2}\right)t$

$= \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} c\left(1 - \frac{v^2}{c^2}\right)t = \sqrt{1 - \frac{v^2}{c^2}} \cdot ct$

Rubiks gruppe

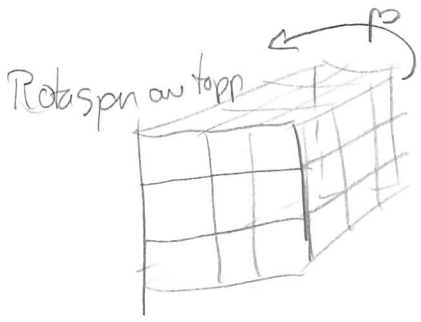


Symmetrier generert av

$\rho = (1234) \quad \mu = (1243)$

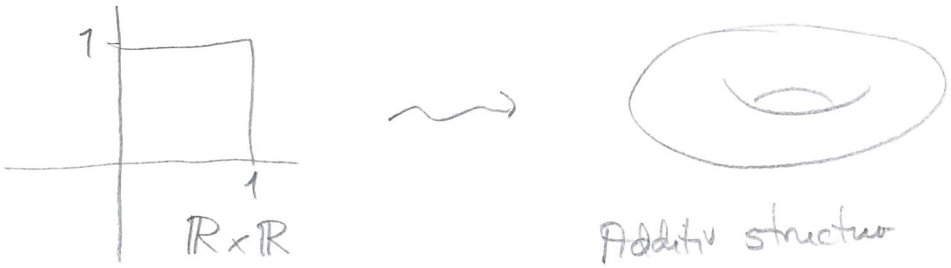
F.eks $(1423) = (1234)(1243)(1432)$

$\rho^4 = \mu^4 = e \quad$ + relasjon



$\beta^4 = e. \quad R = \langle \rho, \mu, \beta \mid \text{relasjoner} \rangle$

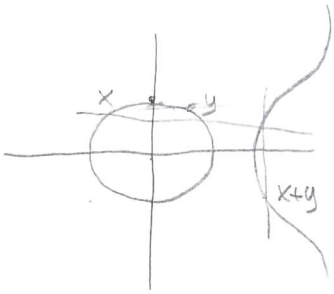
$|R| = 43\,252\,003\,274\,489\,856\,000$



Additiv struktur

Identifiser $(0,y) \sim (1,y)$
$(x,0) \sim (x,1)$

$(x_1,y_1)+(x_2,y_2) = (x_1+x_2, y_1+y_2) \pmod{\mathbb{Z}\times\mathbb{Z}}$



$y^2 = x^3 - x + 1$
Elliptiske kurve

Addisjon: Geometrisk konstruksjon

Identitet: ∞ fjerne punkt i y-retning

Invers: $-(x,y) = (x,-y)$

Linje gjennom (x_1,y_1) (x_2,y_2)

$$\begin{vmatrix} x & y & z \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = (y_1-y_2)x + (x_2-x_1)y + (x_1y_2-y_1x_2)z = 0$$

$\Updownarrow$

sett $z=1$

$$(y_2-y_1)x + (x_1-x_2)y = x_1y_2 - y_1x_2$$

Ex $(0,1)+(1,1)$

Linje: $-y = -1$; $y = 1$

skjæring med kurve $1 = x^3 - x + 1 \Leftrightarrow x^3 - x = 0$

dvs $x=1$ $x=0$ $x=-1$

skjæring: $(-1,1)$

$\Rightarrow (0,1)+(1,1) = (-1,-1)$

Assosiativitet: $\big((0,1)+(1,1)\big)+(0,-1) = (0,1)+\big((1,1)+(0,-1)\big)$

$\|$ $\|$

$(-1,-1)+(0,-1)$ $(0,1)+(3,-5)$

Linje: $y=-1$ $-6x-3y=-3$ eller $2x+y=1$

skjæring $(1,-1)$ $(1-2x)^2 = x^3-x+1$ $\Downarrow$ $y = 1-2x$

sum $(1,1)$ $\Updownarrow$

$x^3 - 4x^2 + 3x = 0$

$x=0$ $\Updownarrow$ $x=3$ $x=1$ $\downarrow$ $y = 1-2 = -1$

skjæring $(1,-1)$

sum $(1,1)$

$G = GL_2(\mathbb{Z}_3) \Rightarrow |\mathbb{Z}_3^4| = 81$

Hvilka har determinant $\neq 0$?

$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc = 0$ dvs $ad = bc$

$a = d = 0$?: $(b,c) = (0,0), (0,1), (0,2), (1,0), (2,0)$
$a \neq 0, d = 0$
$a = 0, d \neq 0$ } $5 \times 5 = 25$ matriser

$a, d \neq 0$. $1 \cdot 1 = 1$, $1 \cdot 2 = 2$, $2 \cdot 1 = 2$, $2 \cdot 2 = 1$ $b, c \neq 0$

$ad = bc = 1$; } 4 matriser
$ad = bc = 2$;

$|G| = 81 - 25 - 2 \cdot 4 = 48 = 2^4 3$

$G_0 = \{ M \in G \mid \det(M) = 1 \}$ $|G_0| = 24$

$G_0 \hookrightarrow G \to G/G_0 \simeq \mathbb{Z}_2$

$g = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$ $g^2 = \begin{pmatrix} 0 & 2 \\ 1 & 2 \end{pmatrix}$ $g^3 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ $g^4 = \begin{pmatrix} 2 & 1 \\ 2 & 0 \end{pmatrix}$ $g^5 = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}$ $g^6 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\det(g) = 1$ $\Rightarrow |g| = 6$ dvs $G_0 \neq S_4$

Abelsk ?

$\begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}$ $\begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix}$

$\begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix} \neq \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix}$ Ikke abelsk

To normale undergrupper 1) $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \right\} \simeq \mathbb{Z}_2$

2) $H = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \right\} \simeq Q$ quaternione gr.

$Q = \langle i, j, k \mid i^2 = j^2 = k^2 = ijk \rangle$

	1	i	j	k
1	1	i	j	k
i	i	-1	k	$-j$
j	j	$-k$	-1	i
k	k	j	$-i$	-1

$ij = k$

$\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}$

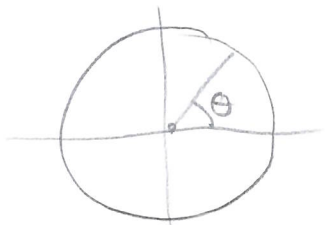
Isomorf $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mapsto 1$ $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \mapsto -1$

$\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} \mapsto i$ $\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} \mapsto -i$

$\begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix} \mapsto j$ $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \mapsto -j$

$\begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix} \mapsto k$ $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \mapsto -k$.

Sirkelgruppa S^1:



$$S_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$S^1 = \langle S_\theta \mid \theta \in [0, 2\pi\rangle \rangle \simeq ([0, 2\pi\rangle, \text{addisjon mod } 2\pi)$$

Abelsk, uendelig generert, undergrupper av alle ordener.