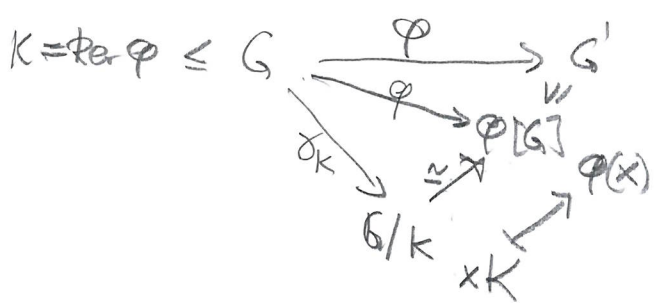
Forelesning 21. februar 2024

Isomorfismeteoremer:



1. isomorfiteorem

Bevis $xK \longmapsto \varphi(x)$ · vel-definert siden

$$\varphi(x) = \varphi(y) \Leftrightarrow y^{-1}x \in K \Leftrightarrow x \in yK$$

symmetri $\Rightarrow y \in xK$.

$\Rightarrow xK = yK$

- 1-1 fordi $\varphi(x) = e'$ betyr $x \in \ker\varphi = K$.
- surjektiv fordi $y \in \varphi[G]$ betyr $\exists x \in G$ s.a. $\varphi(x) = y$; $xK \mapsto y$.

Lemma 34.3

$N \trianglelefteq G$ normal

$\gamma: G \longrightarrow G/N$

$$\varphi: \{N \trianglelefteq H \leq G\} \longrightarrow \{K \trianglelefteq G/N\}$$

(normal) (normal)

$$H \longmapsto \gamma[H]$$

1-1 og på

Bevis

1) $(gN)^{-1}\gamma[H]\,gN = N^{-1}g^{-1}\gamma[H]\,gN = N^{-1}\gamma[g^{-1}Hg]N = N^{-1}\gamma[H]N = \gamma[H]$

2) $N \trianglelefteq H$, $x \in H \Rightarrow xN \subseteq xH = H$.

$\Rightarrow \gamma^{-1}[\varphi(H)] = \gamma^{-1}(\gamma[H]) = N \cdot H = H$.

Det betyr $N \trianglelefteq H_1, H_2$ hvor $\gamma[H_1] = \gamma[H_2]$

Så er $H_1 = \gamma^{-1}(\gamma[H_1]) = \gamma^{-1}(\gamma[H_2]) = H_2$ dus 1-1.

3) La $K \trianglelefteq G/N \Rightarrow \gamma^{-1}(K) \trianglelefteq G$. Vi har $N \trianglelefteq \gamma^{-1}(K)$

(normal) (normal)

Som betyr $\varphi(\gamma^{-1}(K)) = \gamma(\gamma^{-1}(K)) = K$. dus På

QED

$$\begin{array}{ccc} G & \longrightarrow & G/N \\ \text{VI} & & \text{VI} \\ H & \longrightarrow & \curlyvee[H] \\ \text{VI} & & \text{VI} \\ N & \longrightarrow & \{e\} \end{array}$$

Notasjon: $H, N \leq G$

$$HN = \{h \cdot n \mid h \in H, n \in N\}$$

$H \vee N$: Minste undergruppe som inneholder HN

Lemma 34.4.

i) $N \overset{\text{normal}}{\trianglelefteq} G \Rightarrow H \vee N = NH = HN$

$H \leq G$

ii) $H \leq G \Rightarrow NH \leq G$
normal normal

Bevis i) Skal vise at $HN \leq G$.

- $h_1n_1, h_2n_2 \in HN$, $N \underset{\text{normal}}{\Rightarrow} n_1h_2 = h_2n_3$

$$(h_1n_1)(h_2n_2) = h_1(n_1h_2)n_2$$
$$= h_1(h_2n_3)n_2 = h_1h_2 \cdot n_3n_2 \in HN$$

- $e \cdot e = e \in HN$
- $h \cdot n \in HN \Rightarrow (hn)^{-1} = n^{-1}h^{-1} = h^{-1}n_3 \in HN$

Tilsvarende for NH.

ii) $H \underset{\text{normal}}{\leq} G$, $g \in G$.

$$g^{-1}HNg = g^{-1}Hgg^{-1}Ng = HN.$$

QED

Thm 34.5 (2. isomorfisteorem)

$H \leq G \Rightarrow HN/N = H/H\cap N$

N normal

Bevis

- $G \xrightarrow{\gamma} G/N$

  $H \xrightarrow{\gamma|_H \text{ på}} \gamma[H]$ $\Rightarrow \ker(\gamma|_H) = \{x \in H \mid x \in N\}$

  $= H \cap N$

  $\Rightarrow \gamma[H] \simeq H/H\cap N$

- $\gamma|_{HN} : HN \longrightarrow G/N$

  $\gamma|_{HN}(h\cdot n) = \gamma(h)\cdot\gamma(n) = \gamma(h)$

  $\Rightarrow \gamma[HN] = \gamma[H]$

  D.v.s $\gamma|_{HN} : HN \longrightarrow \gamma[H]$

  $\ker(\gamma_{HN}) = \{h\cdot n \in HN \mid \gamma(hn) = \gamma(h) = e\}$

  $= NN = N$

  $\Rightarrow \gamma[H] \simeq HN/N$ Q.E.D.

Ex. $H = \mathbb{Z}\times\mathbb{Z}\times\{0\} \leq \mathbb{Z}\times\mathbb{Z}\times\mathbb{Z} = G$, $N = \{0\}\times\mathbb{Z}\times\mathbb{Z} \leq G$

$\Rightarrow H\cap N = \{0\}\times\mathbb{Z}\times\{0\}$

$H\cdot N = \mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}$

$\Rightarrow HN/N \simeq \mathbb{Z}$ $H/H\cap N \simeq \mathbb{Z}$

Thm 34.7

$K \leq H \leq G$ normale i G. $\Rightarrow G/H \simeq (G/K)/(H/K)$

Bevis

$G \xrightarrow{\varphi} (G/K)/(H/K)$

$g \xmapsto{\varphi} (gK)(H/K)$

- opplagt surjektiv

- $\varphi(g_1 g_2) = (g_1 g_2 K)(H/K)$
$= (g_1 K)(g_2 K)(H/K)$
$= (g_1 K)(H/K) \cdot (g_2 K)(H/K) = \varphi(g_1)\varphi(g_2)$

- $\varphi(g) = (gK)(H/K) = H/K$ betyr $g \in H$
$\Rightarrow \quad H = \ker\varphi \leq G \xrightarrow{\varphi} (G/K)/(H/K)$
$\Rightarrow \quad G/H \simeq (G/K)/(H/K)$. QED

Ex.
$K = 6\mathbb{Z} \leq H = 2\mathbb{Z} \leq G = \mathbb{Z}$
$\Rightarrow \quad G/H = \mathbb{Z}/2\mathbb{Z} \simeq \mathbb{Z}_2$
$G/K = \mathbb{Z}/6\mathbb{Z} = \{6\mathbb{Z}, 6\mathbb{Z}+1, 6\mathbb{Z}+2, 6\mathbb{Z}+3, 6\mathbb{Z}+4, 6\mathbb{Z}+5\} \simeq \mathbb{Z}_6$
$H/K = 2\mathbb{Z}/6\mathbb{Z} = \{6\mathbb{Z}, 6\mathbb{Z}+2, 6\mathbb{Z}+4\} \simeq 2\mathbb{Z}_6$
$\Rightarrow (G/K)/(H/K) = \mathbb{Z}_6/2\mathbb{Z}_6 \simeq \mathbb{Z}_2$

Ex. $K = \left\{\begin{pmatrix}1&0\\0&1\end{pmatrix}, \begin{pmatrix}2&0\\0&2\end{pmatrix}\right\} \leq SL_2(\mathbb{Z}_3)$

$N = \left\{\begin{pmatrix}1&0\\0&1\end{pmatrix}, \begin{pmatrix}2&0\\0&2\end{pmatrix}, \begin{pmatrix}0&2\\1&0\end{pmatrix}, \begin{pmatrix}0&1\\2&0\end{pmatrix}, \begin{pmatrix}2&2\\2&1\end{pmatrix}, \begin{pmatrix}1&1\\1&2\end{pmatrix}, \begin{pmatrix}1&2\\2&2\end{pmatrix}, \begin{pmatrix}2&1\\1&1\end{pmatrix}\right\} = Q \leq SL_2(\mathbb{Z}_3)$

$K \leq N \leq SL_2(\mathbb{Z}_3)$
normale

$SL_2(\mathbb{Z}_3)/N \simeq \mathbb{Z}_3$

$SL_2(\mathbb{Z}_3)/K$

$N/K = \left\{K, \begin{pmatrix}0&2\\1&0\end{pmatrix}K, \begin{pmatrix}2&2\\2&1\end{pmatrix}K, \begin{pmatrix}1&2\\2&2\end{pmatrix}K\right\}$

$\begin{pmatrix}0&2\\1&0\end{pmatrix}^2 = \begin{pmatrix}2&0\\0&2\end{pmatrix} \in K \quad \begin{pmatrix}2&2\\2&1\end{pmatrix}^2 = \begin{pmatrix}2&0\\0&2\end{pmatrix} \in K \quad \begin{pmatrix}1&2\\2&2\end{pmatrix}^2 = \begin{pmatrix}2&0\\0&2\end{pmatrix} \in K$

$\begin{pmatrix}0&2\\1&0\end{pmatrix}\begin{pmatrix}2&2\\2&1\end{pmatrix} = \begin{pmatrix}1&2\\2&2\end{pmatrix}$ dvs $H/K \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$

$\ker\beta \simeq \mathbb{Z}_2 \times \mathbb{Z}_2 \leq SL_2(\mathbb{Z}_3)/K \xrightarrow{\beta} \mathbb{Z}_3$

Ikke oddsh, dvs $SL_2(\mathbb{Z}_3)/K \not\simeq \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$

$$H = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 1 & 2 \end{pmatrix} \right\} \simeq \mathbb{Z}_3$$

$|HN| = 24$ fordi $\begin{pmatrix} 2 & 1 \\ 2 & 0 \end{pmatrix} N \cap N = \emptyset$ $\begin{pmatrix} 2 & 1 \\ 2 & 0 \end{pmatrix} \notin N$

$\Rightarrow HN = SL_2(\mathbb{Z}_3)$

$HN/N \simeq H/N \cap H = H/\{e\} \simeq H$

Betrakt $H/K = \{ K, \begin{pmatrix} 2 & 1 \\ 2 & 0 \end{pmatrix} K, \begin{pmatrix} 0 & 2 \\ 1 & 2 \end{pmatrix} K \}$

$N/K = \{ K, \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} K, \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix} K, \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix} K \}$

$H/K \cdot N/K = \{ K, \begin{pmatrix} 2 & 1 \\ 2 & 0 \end{pmatrix} K, \begin{pmatrix} 0 & 2 \\ 1 & 2 \end{pmatrix} K$

$\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} K, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} K \begin{pmatrix} 2 & 0 \\ 2 & 2 \end{pmatrix} K$

$\begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix} K, \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix} K \begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix} K$

$\begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix} K, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} K, \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix} K \} \simeq SL_2(\mathbb{Z}_3)/K$

$a = \begin{pmatrix} 2 & 1 \\ 2 & 0 \end{pmatrix} K \quad a^3 = e$

$b = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} K \quad b^2 = e$

$c = \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix} K \quad c^2 = e$

$bc = \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix} K = cb$

$ba = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} K = abc$

$ca = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix} K = ab$

dvs $SL_2(\mathbb{Z}_3) = \langle a, b, c \mid a^3 = b^2 = c^2, bc = cb, ba = abc, ca = ab \rangle$

$cac = b$

21. Februar 2024

Litt motivasjon

Simpel gruppe: Gruppe uten ikke-trivielle ekte normale undergrupper

ATOMENE

Kan vi bygge opp alle grupper (endelige) ved hjelp av simple grupper?

Viktigste verktøy:

$$H \leq G \longrightarrow G/H$$

H: normal undergr. — G/H: faktorgruppe

Vi sier at G er en EKSTENSJON av G/H med H.

Ex.

$$\mathbb{Z}_3 \leq S_3 \longrightarrow S_3/\mathbb{Z}_3 \simeq \mathbb{Z}_2$$

så S_3 er bygget opp av $\mathbb{Z}_2$ og $\mathbb{Z}_3$

$$\mathbb{Z}_3 \leq \mathbb{Z}_6 \longrightarrow \mathbb{Z}_6/\mathbb{Z}_3 \simeq \mathbb{Z}_2$$

$\mathbb{Z}_6$ er også bygget opp av $\mathbb{Z}_2$ og $\mathbb{Z}_3$

Forskjellen? Husk $\mathbb{Z}_3 = \langle g \rangle$, $\mathbb{Z}_2 = \langle h \rangle$

så gir $gh = hg$ $\mathbb{Z}_6$ } enste mulighet

mens $gh = hg^{-1}$ gir S_3

Ex

$$K \leq H \leq SL_2(\mathbb{Z}_3)$$

K: $\mathbb{Z}_2$ — H: quaternioner

$$SL_2(\mathbb{Z}_3)/H \simeq \mathbb{Z}_3 \quad H/K \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$$

$\Rightarrow SL_2(\mathbb{Z}_3)$ er bygget opp av $\mathbb{Z}_2, \mathbb{Z}_2$ og $\mathbb{Z}_3$

En viktig användelse:

$$\{e\} = G_0 \leq \cdots \leq G_{m-1} \leq G_m = G$$

G_{m-1}: normala undergrupp (maximal hyde)

G: Galois gruppa till rötteroppen till ett irreducibelt polynom

betyr at G er byggd opp av G_j/G_{j-1} (simpel pga maximalitet)

Hvis alle G_j/G_{j-1} er abelske så kan vi finne en formel for rotten til polynomet

För en generell 5. gradsekvation är $G = S_5$

$$\{e\} \leq A_5 \leq S_5$$

maximal hyde av största undergr.

$$S_5/A_5 \simeq \mathbb{Z}_2 \quad , \quad A_5/\{e\} = A_5$$

ikke abelsk

$\Rightarrow$ Abel-Ruffini-teoremet!