Forelesning 10 april 2024

Husk

$F \leq E \ni \alpha$, $\beta \in E$, $\operatorname{irr}(\alpha, F) = \operatorname{irr}(\beta, F)$ (konjugerte) $\Rightarrow \exists\, \psi_{\alpha,\beta} : F(\alpha) \xrightarrow[\text{iso}]{\cong} F(\beta)$

Hvis alle røttene ligger i samme kropp E, så vil alle isomorfier $\psi_{\alpha,\beta}$ være automorfier.

Derfor:

Defn. 50.1.

$F \leq \overline{F}$ alg. tilhukning

$M = \{f_i(x) \in F[x] \mid i \in I\}$

$F \leq E$ kalles <u>rotkroppen</u> til M over F; $E = F(M)$

Den minste kroppen E som inneholder alle røtter i M

En kropp $F \leq E$ kalles en rotkropp dersom $\exists\, M$ slik at $E = F(M)$.

Ex. $\mathbb{Q}(\sqrt{2}, \sqrt{3}) = \mathbb{Q}(x^2-2, x^2-3) = \mathbb{Q}(x^4-5x^2+6)$

F vl E vl F

E er en rotkropp dersom enhver automorfi av F induserer en automorfi av E.

$f(x) = (x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n)$ ← polynomet splitter

Cor 50.6

$F \leq E$ rotkropp, $f(x) \in F[x]$ slik at $\exists\, \alpha \in E$ med $f(\alpha) = 0$ $\Rightarrow$ $f(x)$ splitter i E.

Cor. 50.7

$F \leq E$ rotthropp $\Rightarrow \{E:F\} = |G(E/F)|$

Bevis

$$\begin{array}{ccc} E & \overset{\simeq}{\Rightarrow} & E' \\ \text{VI} & & \text{VI} \\ F & = & F \end{array} \qquad \begin{array}{c} \overline{F} \\ \text{VI} \\ \overline{E} \end{array}$$

$E \Rightarrow E' = E$ dvs alle rotter i $\overline{E}$

Rotthropp

$\{E:F\}$ = antall isomorfi $E \to E'$

= antall automorfi $E \to E = |G(E/F)|$

$\Rightarrow \{:F\} = |G(E/F)|$

— o — o —

3 gradsligning $x^3 + px + q = 0$

Løsninger

$$x_1 = R^{\frac{1}{3}} - \frac{p}{3R} R^{\frac{2}{3}}$$

$$x_2 = \omega R^{\frac{1}{3}} - \omega^2 \frac{p}{3R} R^{\frac{2}{3}}$$

$$x_3 = \omega^2 R^{\frac{1}{3}} - \omega \frac{p}{3R} R^{\frac{2}{3}}$$

Merk. $x_3 = -x_1 - x_2$

$x_1 \in \langle x_2^2, x_2, 1 \rangle$

hvor $R = \frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}$ og $\omega^3 = 1$

$$\mathbb{Q} \leq \mathbb{Q}(R) \leq \mathbb{Q}(x^3+px+q) = \mathbb{Q}\left(\omega R^{\frac{1}{3}} - \omega^2 \frac{p}{3R} R^{2/3}\right)$$

Rotthropp

$$G\left(\mathbb{Q}(R)/\mathbb{Q}\right) \simeq \mathbb{Z}_2$$

$$G\left(\mathbb{Q}(x^3+px+q)/\mathbb{Q}(R)\right) \simeq \mathbb{Z}_3 = \langle t \rangle$$

hvor $t : x_1 \mapsto x_2$

Løsbar ved radikaluttrykk

$$\mathbb{Q} \leq E_1 \leq E_2 \leq \ldots \leq \mathbb{Q}(x^3+px+q)$$

Rotthropp

$$G\left(F_{j+1}/E_j\right) \simeq \mathbb{Z}_p$$

Betrakter mengden av restklasser

$$\{\tau \cdot G(K/E) \mid \tau \in G(K/F)\}$$

K
|
E
|
F

$\tau \cdot G(K/E) = \sigma\, G(K/E) \iff \tau^{-1}\sigma \in G(K/E)$

dvs $\tau^{-1}\sigma(\alpha) = \alpha \quad \forall \alpha \in E$

elle $\tau(\alpha) = \sigma(\alpha) \quad \forall \alpha \in E$

og vice versa

Venstre restklasse $\tau \cdot G(K/E)$ svarer til

$$\begin{array}{ccc} K & \xrightarrow{\tau} & K \\ | & & | \\ E & \xrightarrow[\cong]{\tau|_E} & E' \\ | & & | \\ F & \xrightarrow{=} & F \end{array}$$

$G(K/E) \leq G(K/F) \iff \forall \sigma \in G(K/E),\ \tau \in G(K/F)\ ;\ \tau^{-1}\sigma\tau \in G(K/E)$

normal undergruppe

dvs $\forall \alpha \in E \quad \tau^{-1}\sigma\tau(\alpha) = \alpha$

$\Downarrow$

$\sigma(\tau(\alpha)) = \tau(\alpha)$

$\Downarrow$

$\tau(\alpha) \in E$

som betyr $E' = E$

og E er en rotthopp $/F$

Viktig $E = K_{G(K/E)}$

1) $E \leq K_{G(K/E)}$ opplagt

2) $K_{G(K/E)} \leq E$

$\alpha \in K, \alpha \notin E$; K rotthopp

betyr $\exists \psi : K \to K,\ \psi|_E = id$

$\operatorname{Irr}(\psi(\alpha), F) = \operatorname{Irr}(\alpha, F),\ \psi(\alpha) \neq \alpha$

$\Rightarrow \alpha \notin K_{G(K/E)}$

$$K_{G(K/E)} = \{\alpha \in K \mid \forall \psi : K \to K,\ \psi|_E = id\ ;\ \psi(\alpha) = \alpha\}$$

Radikal-tårn:

$$F \leq E_1 \leq E_2 \leq \dots \leq E$$

$$E_{j+1} = E_j\left(\sqrt[n]{q_j}\right) \Rightarrow G\left(E_{j+1}/E_j\right) \simeq \mathbb{Z}_n$$

Galois-teori:

rotfrapp/F K ⊥ E ⊥ F

$G(K/F)$ ⊥ $G(K/E)$ normal undergrupp

$$G(K/F)/G(K/E) \simeq G(E/F)$$

En likning $f(x)=0$ kan løses ved rotutdragning $\Leftrightarrow$ $\exists$ radikal-tårn $F \leq E_1 \leq E_2 \leq \dots \leq E_n \leq K = F(f(x))$ Rotfrappen

$$\Updownarrow$$

$$G(K/K) \leq G(K/E_n) \leq \dots \leq G(K/E_1) \leq G(K/F)$$

Ex $f(x) = x^3 - 2$

Røtter $\sqrt[3]{2} = \alpha,\ \omega\alpha,\ \omega^2\alpha$

$\omega^3 = 1$

$K = \mathbb{Q}(\alpha, \omega)$

$$\mathbb{Q} \leq \mathbb{Q}(\omega) \leq \mathbb{Q}(\alpha,\omega)$$

$$\Updownarrow$$

$$G\left(\mathbb{Q}(\alpha,\omega)/\mathbb{Q}(\omega)\right) \leq G\left(\mathbb{Q}(\alpha,\omega)/\mathbb{Q}\right) \leq G\left(\mathbb{Q}(\alpha,\omega)/\mathbb{Q}\right)$$

$\lhd$ $\lhd$ $\lhd$

$$\{e\} \leq \mathbb{Z}_3 \leq S_3$$

$$\mathbb{Z}_3/\{e\} \simeq \mathbb{Z}_3 = \langle\tau\rangle \qquad S_3/\mathbb{Z}_3 \simeq \mathbb{Z}_2$$

$\lhd$ $\lhd$

$G(\mathbb{Q}(\alpha,\omega)/\mathbb{Q}(\omega))$ $G(\mathbb{Q}(\omega)/\mathbb{Q})$

$\mathbb{Q}(\alpha,\omega) \xrightarrow{\sim} \mathbb{Q}(\alpha,\omega)$

⊥ ∥

$\mathbb{Q}(\omega)$

$\mathrm{id} : \alpha \mapsto \alpha$

$\tau : \alpha \mapsto \omega\alpha$

$\tau^2 : \alpha \mapsto \omega^2\alpha$