Forelesning 15. april 2024

Galois teori:

1.
$$\begin{array}{c} \bar{F} \supseteq \beta \\ | \\ \alpha \in E \\ | \\ F \end{array} \qquad \alpha, \beta \text{ konjugerte/F} \iff irr(\alpha, F) = irr(\beta, F) \Rightarrow \begin{array}{ccc} F(\alpha) & \xrightarrow[\simeq]{\exists \psi : \alpha \to \beta} & F(\beta) \\ | & & | \\ F & = & F \end{array}$$

2.
$$\begin{array}{ccc} \bar{F} & \xrightarrow[\simeq]{\sigma} & \bar{F} \\ | & & | \\ \alpha \in E & \xrightarrow{\sigma|_E} & E' \\ | & & | \\ F & = & F \end{array} \qquad irr(\sigma(\alpha), F) = irr(\alpha, F)$$

3. E, F delkropper av $\bar{F}$

$G(E/F) = \left\{ \begin{array}{ccc} E & \xrightarrow{\simeq} & E \\ | & & | \\ F & = & F \end{array} \right\}$ gruppe

1) $S \subseteq G(E/F)$

$\Downarrow$

$E_S = \{ \alpha \in E \mid \psi(\alpha) = \alpha \ \forall \psi \in S \}$ kropp

ii) $F \leq E_{G(E/F)}$

4. $\begin{array}{c} \bar{F} \\ | \\ E \\ | \\ F \end{array}$

rotkropp: $E \xrightarrow[\simeq]{\psi} E'$, $F = F$ $\Rightarrow E' = E$, ψ : automorfi

$\Rightarrow |G(E/F)| = \{E : F\}$

5. E, F $\Rightarrow [E : F] = \{E : F\}$ (alle røtter har multiplisitet 1)

endelig (separabel)

6. E rotkropp (separabel) $[E : F] = \{E : F\} = |G(E/F)|$

Ex $x^3-2 \in \mathbb{Q}[x]$ Rötter $\sqrt[3]{2}, \sqrt[3]{2}\omega, \sqrt[3]{2}\omega^2$, $\omega^3=1, \omega \neq 1$.

$$\begin{array}{ccc} \mathbb{Q}(\sqrt[3]{2}) & \xrightarrow{\simeq} & \mathbb{Q}(\sqrt[3]{2}\cdot\omega) \\ | & & | \\ \mathbb{Q} & = & \mathbb{Q} \end{array}$$

$$\begin{array}{ccc} \mathbb{Q}(\sqrt[3]{2}) & \xrightarrow{\varphi \simeq} & E \\ | & & | \\ \mathbb{Q} & = & \mathbb{Q} \end{array}$$

betyr $\operatorname{irr}(\varphi(\sqrt[3]{2}), \mathbb{Q}) = x^3-2$

dvs. $E = \mathbb{Q}(\sqrt[3]{2})$
eller $E = \mathbb{Q}(\sqrt[3]{2}\cdot\omega)$
eller $E = \mathbb{Q}(\sqrt[3]{2}\cdot\omega^2)$

$G(\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}) \simeq \{id\}$ $|G(\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q})| = 1$

$[\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}] = 3$ Basis $\{1, \sqrt[3]{2}, \sqrt[3]{4}\}$

$\{\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}\} = 3$ (se over)

Rotthrapp $\mathbb{Q}(\sqrt[3]{2}, \sqrt[3]{2}\omega, \sqrt[3]{2}\omega^2) = \mathbb{Q}(\sqrt[3]{2}, \omega)$

$G = G(\mathbb{Q}(\sqrt[3]{2}, \omega)/\mathbb{Q}) \simeq S_3$ $|G| = 6$

$[\mathbb{Q}(\sqrt[3]{2}, \omega) : \mathbb{Q}] = 6$ Basis $\{1, \sqrt[3]{2}, \sqrt[3]{4}, \omega, \sqrt[3]{2}\omega, \sqrt[3]{4}\omega\}$

$\{\mathbb{Q}(\sqrt[3]{2}, \omega) : \mathbb{Q}\} = 6$

$\sqrt[3]{2} \mapsto \sqrt[3]{2}$ $\omega \mapsto \omega$
$\sqrt[3]{2}\omega$ $\omega \mapsto \omega^2$.
$\sqrt[3]{2}\omega^2$

Oppgave 3 - 2007

$$\alpha = \sqrt{5 - 2\sqrt{5}}$$

a) $\text{irr}(\alpha, \mathbb{Q})$:

$$\alpha^2 = 5 - 2\sqrt{5}$$

$$(\alpha^2 - 5)^2 = (-2\sqrt{5})^2$$

$$\alpha^4 - 10\alpha^2 + 25 = 4 \cdot 5 = 20$$

$$\Rightarrow \quad \alpha^4 - 10\alpha^2 + 5 = f(x)$$

Sette $\beta = \alpha^2$; $\beta^2 - 10\beta + 5 = 0$

$$\Rightarrow \beta = \frac{10 \pm \sqrt{100 - 20}}{2} = 5 \pm 2\sqrt{5}$$

$$\Rightarrow \quad \alpha^4 - 10\alpha^2 + 5 = (\alpha^2 - (5 + 2\sqrt{5}))(\alpha^2 - (5 - 2\sqrt{5}))$$

og ingen lineære faktorer

b) $K = \mathbb{Q}(f(x))$ Røttene er gitt ved $\pm\sqrt{5 - 2\sqrt{5}}, \pm\sqrt{5 + 2\sqrt{5}}$

$$\pm s, \pm t$$

$$s^2 = 5 - 2\sqrt{5} \qquad t^2 = 5 + 2\sqrt{5}$$

$$\Rightarrow s^2 + t^2 = 10$$

$$st = \sqrt{5 - 2\sqrt{5}} \cdot \sqrt{5 + 2\sqrt{5}}$$

$$= \sqrt{5^2 - 4 \cdot 5} = \sqrt{5} = \frac{5 - s^2}{2}$$

$$\Rightarrow [K : \mathbb{Q}] = 4 \quad \text{Basis } \{1, s, t, \sqrt{5}\}$$

og hvis $\sqrt{5} = a + bs + ct$, $a, b, c \in \mathbb{Q}$, $s \neq 0$

$$5 = a^2 + b^2 s^2 + c^2 t^2 + 2ab\,s + 2act + 2bc\,st$$

dvs $(-2b^2 + 2c^2 + 2bc)\sqrt{5}$

$$+ 2ab\sqrt{5 - 2\sqrt{5}} + 2ac\sqrt{5 + 2\sqrt{5}}$$

1) $a = 0 \quad c^2 + bc - b^2 = 0 \Rightarrow c = \frac{-b \pm \sqrt{b^2 + 4b^2}}{2} = \frac{b}{2}(-1 \pm \sqrt{5})$

2) $b = c = 0$

Vanskelig $a = \sqrt{5} \in \mathbb{Q}$

Motsigelse

$$[K:\mathbb{Q}(\sqrt{5})] = \frac{[K:\mathbb{Q}]}{[\mathbb{Q}(\sqrt{5}):\mathbb{Q}]} = \frac{4}{2} = 2$$

$[\mathbb{Q}(\sqrt{5}):\mathbb{Q}] = 2$ Basis $\{1, \sqrt{5}\}$

c) $G(K/\mathbb{Q}) \simeq \mathbb{Z}_4$ Vi vet $|G(K/\mathbb{Q})| = 4$ (Rotkropp)

$$\{id\} \leq G(K/\mathbb{Q}(\sqrt{5})) \leq G(K/\mathbb{Q})$$
$$\mathbb{Z}_2 \qquad \mathbb{Z}_2$$

$$G(\mathbb{Q}(\sqrt{5})/\mathbb{Q}) \simeq G(K/\mathbb{Q}) / G(K/\mathbb{Q}(\sqrt{5}))$$

eller

$$\begin{array}{ccccc} G(K/\mathbb{Q}(\sqrt{5})) & \hookrightarrow & G(K/\mathbb{Q}) & \longrightarrow & G(\mathbb{Q}(\sqrt{5})/\mathbb{Q}) \\ \wr & & \downarrow \psi & & \wr \\ \mathbb{Z}_2 & & & & \mathbb{Z}_2 \end{array}$$

$$\text{irr}(\alpha, \mathbb{Q}(\sqrt{5})) = x^2 - 5 + 2\sqrt{5} = (x - \sqrt{5-2\sqrt{5}})(x + \sqrt{5-2\sqrt{5}})$$

dvs

$$\begin{array}{ccc} K & \xrightarrow{\psi} & K \\ \vee & & \vee \\ \mathbb{Q}(\sqrt{5}) & \longrightarrow & \mathbb{Q}(\sqrt{5}) \\ \vee & & \vee \\ \mathbb{Q} & = & \mathbb{Q} \end{array}$$

$\psi(\sqrt{5-2\sqrt{5}}) = \sqrt{5+2\sqrt{5}}$

$\psi(\sqrt{5-2\sqrt{5}}^2) = \psi(5-2\sqrt{5})$

$= 5 - 2\psi(\sqrt{5})$

$= 5 + 2\sqrt{5}$

dvs $\psi|_{\mathbb{Q}(\sqrt{5})} : \sqrt{5} \mapsto -\sqrt{5}$

$\Rightarrow \deg(\psi) = 4$

$$\mathbb{Z}_2 \hookrightarrow \mathbb{Z}_4 \longrightarrow \mathbb{Z}_2$$
$$1 \longmapsto 2 \longmapsto 0$$
$$1 \longmapsto 1$$

eller

$$\mathbb{Z}_2 \hookrightarrow \mathbb{Z}_2 \times \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2$$
$$1 \longmapsto (1,0)$$
$$(0,1) \longmapsto 1$$

Oppgave 3 - 2008

$$f(x) = x^4 - 2x^2 - 3$$

a) ① $(f(x))$: $\quad f(x) = 0 \Rightarrow x^2 = \frac{2 \pm \sqrt{4+12}}{2} \begin{cases} 3 \\ -1 \end{cases}$

$$\Rightarrow f(x) = (x^2-3)(x^2+1)$$

$$\Rightarrow ① (f(x)) = ℚ(\sqrt{3}, i)$$

$$[① (f(x)) : ℚ] = [ℚ(f(x)) : ℚ(\sqrt{3})] \cdot [ℚ(\sqrt{3}) : ℚ]$$

Basis $\{1, i\}$

$$= 2 \cdot 2 = 4$$

$$G(ℚ(f(x))/ℚ) \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$$

b) $\quad ℚ(f(x)) = ℚ(\alpha) \qquad \alpha = \sqrt{3} + i$

$$\Rightarrow \alpha^2 = 3 + 2\sqrt{3}i - 1 = 2 + 2\sqrt{3}i$$

$$\alpha^3 = (\sqrt{3}+i)(2+2\sqrt{3}i)$$

$$= 2\sqrt{3} + 6i + 2i - 2\sqrt{3} = 8i$$

$$ℚ(\alpha) = ℚ(\sqrt{3}+i) = ℚ(\sqrt{3}, i)$$

fordi $\quad i = \frac{\alpha^3}{8}, \quad \sqrt{3} = \alpha - \frac{\alpha^3}{8}, \quad \sqrt{3}i = \frac{\alpha^2 - 2}{2}$

$$\alpha^4 = 4 + 8\sqrt{3}i - 12 = -8 + 8\sqrt{3}i$$

$$\Rightarrow \alpha^4 - 4\alpha^2 + 16 = 0 \Rightarrow \text{irr}(\alpha, ℚ) = x^4 - 4x^2 + 16$$

irreduserbelt fordi $\quad x^2 = \frac{4 \pm \sqrt{16-64}}{2} = 2 \pm 2\sqrt{3}i$

Def. 53.1

K ändlig normal utvidgning av F $\Leftrightarrow$ K ostruppl/F

K vl F

Thm 53.2

K end. normal utv. av F $\Rightarrow$

i) K normal utvidgn av E

ii) $G(K/E) = \{\psi : K \to K \mid \psi|_E = id\} \leq G(K/F)$

iii) $\sigma, \tau \in G(K/F)$ uppf $\sigma|_E = \tau|_E$
$\Leftrightarrow$ $\sigma\, G(K/E) = \tau\, G(K/E)$

K vl E vl F

Bevis i) $K = F(f_1(x), \dots, f_n(x))$ $f_i(x) \in F[x]$
$\Rightarrow f_i(x) \in E[x]$

ii) $\psi \in G(K/E) \Rightarrow \psi \in G(K/F)$
dvs $G(K/E) \leq G(K/F)$

iii) $\sigma\, G(K/E) = \tau\, G(K/E)$
$\Updownarrow$
$\tau^{-1}\sigma \in G(K/E)$
$\Updownarrow$
$\tau^{-1}\sigma(\alpha) = \alpha \quad \forall \alpha \in E$
$\Updownarrow$
$\sigma(\alpha) = \tau(\alpha) \quad \forall \alpha \in E.$

Defn 53.5

K efld. normal utv. av F $\Leftrightarrow$ $G(K/F)$ Galoisgruppa till K över F

K vl F

Thm 53.6 (Galois)

$$\begin{array}{l} K \text{ normal/} F \\ \text{vi} \\ E \\ \text{vi} \\ F \end{array} \qquad \lambda(E) \leq G(K/F), \quad \lambda(E) = \{\Psi \in G(K/F) \mid \Psi|_E = id\}$$

$$\Rightarrow \left\{ \begin{array}{c} K \\ \text{vi} \\ E \\ \text{vi} \\ F \end{array} \right\} \overset{1\text{-}1}{\longleftrightarrow} \left\{ \begin{array}{c} G(K/F) \\ \text{vi} \\ \lambda(E) \end{array} \right\}$$

Slik at

1) $\lambda(E) = G(K/E)$

2) $E = K_{G(K/E)} = K_{\lambda(E)}$

3) $H \leq G(K/F) \Rightarrow \lambda(K_H) = H$

4) $[K:E] = |\lambda(E)|$ og $[E:F] = (G(K/F) : \lambda(E))$

5) $\begin{array}{l} E \\ \text{vi} \\ F \\ \text{normal} \end{array} \Leftrightarrow \lambda(E) \leq G(K/F)$ normal

$$\Downarrow$$

$$G(E/F) \cong G(K/F)/G(K:E)$$

6) Korrespondansen reverserer ordning

Bevis

1) Definisjon

2) $E \leq K_{G(K/E)}$ opplagt

$K_{G(K/E)} \leq E$ La $\alpha \in K \setminus E$. Siden K er normal/E $\exists \Psi : K \to K$ slik at $\Psi(\alpha) \neq \alpha$ er rot i $\text{irr}(\alpha, F)$ og $\Psi|_E = id$

$\Rightarrow \alpha \notin K_{G(K/E)}$.

4) Telle restklasser

3) og 5) Utelates

Ex $f(x) = x^3 - 2$

$$\begin{array}{ccc} & \mathbb{Q}(\sqrt[3]{2}, i) = K & \\ \swarrow & & \searrow \\ \mathbb{Q}(i) & & \mathbb{Q}(\sqrt[3]{2}) \\ \searrow & & \swarrow \\ & \mathbb{Q} & \end{array}$$

$$G(K/\mathbb{Q}) \simeq S_3 = \langle \rho, \mu \mid \rho^3 = \mu^2 = id,\ \rho\mu = \mu\rho^2 \rangle$$

$$\begin{array}{ccc} \langle \rho \rangle = \mathbb{Z}_3 \simeq G(K/\mathbb{Q}(i)) & & G(K/\mathbb{Q}(\sqrt[3]{2})) \simeq \mathbb{Z}_2 = \langle \mu \rangle \\ \text{(normal)} \downarrow & & \downarrow \text{(itzke normal)} \\ & G(K/K) & \\ & \| & \\ & \{id\}_3 & \end{array}$$

$$G(K/\mathbb{Q}) / G(K/\mathbb{Q}(i)) \simeq G(\mathbb{Q}(i)/\mathbb{Q}) \simeq \mathbb{Z}_2$$

$$\left(G(K/\mathbb{Q}) : G(K/\mathbb{Q}(\sqrt[3]{2}))\right) = 3$$

Resthlass $G(K/\mathbb{Q}(\sqrt[3]{2}))$, $\psi\, G(K/\mathbb{Q}(\sqrt[3]{2}))$

$\psi : \omega \mapsto \omega^2$