Forelesning 24. april 2024

Ex $f(x) = x^5 - 1 = (x-\zeta)(x-\zeta^4)(x-\zeta^2)(x-\zeta^3)(x-1)$
$= x^4 + x^3 + x^2 + x + 1$

$$\zeta = \frac{\sqrt{5}-1}{4} + i\frac{\sqrt{10+2\sqrt{5}}}{4}$$
$$\zeta^4 = \frac{\sqrt{5}-1}{4} - i\frac{\sqrt{10+2\sqrt{5}}}{4}$$
$$\zeta^2 = \frac{5-2\sqrt{5}+1}{16} + \frac{\sqrt{5}-1}{2} i\frac{\sqrt{10+2\sqrt{5}}}{4} - \frac{10+2\sqrt{5}}{16}$$
$$= \frac{-4-4\sqrt{5}}{16} + i\frac{\sqrt{6-2\sqrt{5}}\sqrt{10+2\sqrt{5}}}{8}$$
$$= -\frac{\sqrt{5}+1}{4} + i\frac{\sqrt{40-8\sqrt{5}}}{8} = -\frac{\sqrt{5}+1}{4} + i\frac{\sqrt{10-2\sqrt{5}}}{4}$$

$[\mathbb{Q}(\zeta):\mathbb{Q}] = 4$

$$\zeta + \zeta^4 = \frac{\sqrt{5}-1}{2}$$
$$\zeta - \zeta^4 = i\frac{\sqrt{10+2\sqrt{5}}}{2}$$

$$(\zeta+\zeta^4)(\zeta-\zeta^4) = \zeta^2 - \zeta^3 = \frac{\sqrt{10-2\sqrt{5}}}{2}$$

dvs $\mathbb{Q}(\zeta) = \mathbb{Q}\left(\sqrt{5}, i\sqrt{10+2\sqrt{5}}\right)$

$\mathbb{Q} \leq \mathbb{Q}(\sqrt{5}) \leq \mathbb{Q}(\zeta)$

$[\mathbb{Q}(\sqrt{5}):\mathbb{Q}] = 2 \quad [\mathbb{Q}(\zeta):\mathbb{Q}(\sqrt{5})] = 2$

$$4\zeta - \sqrt{5} + 1 = i\sqrt{10+2\sqrt{5}}$$

$$\Rightarrow 16\zeta^2 + 8\zeta(-\sqrt{5}+1) + 6 - 2\sqrt{5} = -10 - 2\sqrt{5}$$

$$\Rightarrow \zeta^2 + \frac{-\sqrt{5}+1}{2}\zeta + 1 = 0$$

$\beta = i\sqrt{10+2\sqrt{5}}$
$i\sqrt{10-2\sqrt{5}} = \frac{\sqrt{5}-1}{2}\beta$

$G(\mathbb{Q}(\zeta)/\mathbb{Q}(\sqrt{5})) \triangleleft G(\mathbb{Q}(\zeta)/\mathbb{Q}) \longrightarrow G(\mathbb{Q}(\sqrt{5})/\mathbb{Q})$

$\sigma \longmapsto \psi_{\sqrt{5},-\sqrt{5}}$

$\psi_{\beta,-\beta}$ $\quad \sigma: \sqrt{5} \mapsto -\sqrt{5}$

$\sigma(\beta^2) = \sigma(-10-2\sqrt{5}) = -10+2\sqrt{5}$

$\Rightarrow \sigma(\beta) = \pm\frac{\sqrt{5}-1}{2}\beta$

$\Rightarrow \sigma^2(\beta) = \pm\sigma\frac{(\sqrt{5}-1)}{2}\beta = \frac{\sqrt{5}-1}{2}\beta$

$= \frac{-\sqrt{5}-1}{2}\frac{\sqrt{5}-1}{2}\beta$

$= -\beta$

$G(\mathbb{Q}(\zeta)/\mathbb{Q}) \simeq \mathbb{Z}_4 \ni \sigma^2 = \psi_{\beta,-\beta}$

$$G(\mathbb{Q}(\zeta)/\mathbb{Q}) \xrightarrow{\sigma} G(\mathbb{Q}(\zeta)/\mathbb{Q})$$
$$\zeta \longmapsto \zeta^2$$

$$\Rightarrow \quad \sigma^2(\zeta) = \sigma(\zeta^2) = \sigma(\zeta)^2 = (\zeta^2)^2 = \zeta^4$$
$$\sigma^3(\zeta) = \sigma(\sigma^2(\zeta)) = \sigma(\zeta^4) = \sigma(\zeta)^4 = (\zeta^2)^4 = \zeta^3$$
$$\sigma^4(\zeta) = \sigma(\sigma^3(\zeta)) = \sigma(\zeta^3) = \sigma(\zeta)^3 = (\zeta^2)^3 = \zeta$$

$$\Rightarrow \quad \sigma(\sqrt{5}) = \sigma(2(\zeta+\zeta^4)+1)$$
$$= 2\sigma(\zeta) + 2\sigma(\zeta^4) + 1$$
$$= 2\zeta^2 + 2\zeta^3 + 1$$
$$= 2\left(-\frac{\sqrt{5}+1}{4}\right)\cdot 2 + 1 = -\sqrt{5}$$

$$\sigma(\beta) = \sigma\left(i\sqrt{10+2\sqrt{5}}\right) = \sigma(2(\zeta-\zeta^4))$$
$$= 2(\sigma(\zeta) - \sigma(\zeta^4))$$
$$= 2(\zeta^2 - \zeta^3) = i\sqrt{10-2\sqrt{5}}$$

$$G(\mathbb{Q}(\zeta)/\mathbb{Q}) \xrightarrow[\text{Mostr}]{} G(\mathbb{Q}(\sqrt{5})/\mathbb{Q})$$
$$\sigma \longmapsto \psi_{\sqrt{5},-\sqrt{5}}$$

$$G(\mathbb{Q}(\zeta)/\mathbb{Q}(\sqrt{5})) \longrightarrow G(\mathbb{Q}(\zeta)/\mathbb{Q})$$
$$\psi_{\beta,-\beta} \longmapsto \psi: \begin{array}{l} \beta \mapsto -\beta \\ \sqrt{5} \mapsto \sqrt{5} \end{array}$$

Mostr $\quad \sigma^2(\sqrt{5}) = \sigma(\sigma(\sqrt{5})) = \sigma(-\sqrt{5}) = -\sigma(\sqrt{5}) = -(-\sqrt{5}) = \sqrt{5}$

$$\sigma^2(\beta) = \sigma(\sigma(\beta)) = \sigma\left(i\sqrt{10-2\sqrt{5}}\right)$$
$$= \sigma\left(\frac{\sqrt{5}-1}{2}\beta\right) =$$
$$= \frac{\sigma(\sqrt{5})-1}{2}\sigma(\beta)$$
$$= \frac{-\sqrt{5}-1}{2} i\sqrt{10-2\sqrt{5}}$$
$$= -\frac{\sqrt{(\sqrt{5}+1)^2}}{2} i\sqrt{10-2\sqrt{5}}$$
$$= -\frac{i}{2}\sqrt{(6+2\sqrt{5})(10-2\sqrt{5})} = -\frac{i}{2}\sqrt{40+8\sqrt{5}}$$
$$= -i\sqrt{10+2\sqrt{5}} = -\beta$$

Ex $f(x) = x^5 - 2$ $\quad \xi^5 = 1$, $\xi = \frac{\sqrt{5}-1}{4} + i\frac{\sqrt{10+2\sqrt{5}}}{4}$

$K = \mathbb{Q}(f(x))$ $\quad \alpha^5 = 2$, $\alpha \in \mathbb{R}$

Rothrupp

$\mathbb{Q} \leq \mathbb{Q}(\xi) \leq K$ $\quad G(\mathbb{Q}(\xi)/\mathbb{Q}) \simeq \mathbb{Z}_4$

$[K : \mathbb{Q}(\xi)] = 5$ $\quad$ Base $\{1, \xi, \xi^2, \xi^3, \xi^4\}$

$\Rightarrow G(K/\mathbb{Q}(\xi)) \simeq \mathbb{Z}_5$

$$\langle \psi \rangle \cong G(K/\mathbb{Q}(\xi)) \hookrightarrow G(K/\mathbb{Q}) \longrightarrow G(\mathbb{Q}(\xi)/\mathbb{Q}) \cong \langle \sigma \rangle$$

$\simeq \mathbb{Z}_5$ $\qquad\qquad$ $\simeq \mathbb{Z}_4$

$|G(K/\mathbb{Q})| = 20$

$\psi(\alpha) = \xi\alpha$

$\psi(\xi) = \xi$

$\sigma(\xi) = \xi^2$ $\quad \sigma(\xi^2) = \xi^4$ $\quad \sigma(\xi^3) = \xi$

$\bar{\sigma}(\xi) = \xi^2$, $\bar{\sigma}(\alpha) = \alpha$

$\Rightarrow \bar{\sigma}^{-1}\psi\bar{\sigma}(\xi) = \bar{\sigma}^{-1}\psi(\xi^2) = \bar{\sigma}^{-1}(\xi^2) = \xi = \psi^3(\xi)$

$\bar{\sigma}^{-1}\psi\bar{\sigma}(\alpha) = \bar{\sigma}^{-1}\psi(\alpha) = \bar{\sigma}^{-1}(\xi\alpha) = \xi^3\alpha = \psi^3(\alpha)$

$\Rightarrow G(K/\mathbb{Q}) \simeq \langle \bar{\sigma}, \psi \mid \bar{\sigma}^4 = \psi^5 = id,\ \psi\bar{\sigma} = \bar{\sigma}\psi^2 \rangle$

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$\psi = (12345)$ $\quad \psi\bar{\sigma} = (12345)(2354) = (1243)$

$\bar{\sigma} = (2354)$ $\quad \bar{\sigma}\psi^3 = (2354)(14253) = (1243)$

