Eksamen 2014

1. $\rho \in S_n \Rightarrow \rho(1234\ldots n)\rho^{-1} = (\rho(1)\rho(2)\cdots\rho(n))$

a) $(a_1 a_2 \ldots a_n) = \rho(123\ldots n)\rho^{-1}$ for $\rho(i) = a_i$

n-sykel

Anta $\rho_1(123\ldots n)\rho_1^{-1} = \rho_2(123\ldots n)\rho_2^{-1}$

$\Rightarrow (\rho_2^{-1}\rho_1)(12\ldots n)\rho_1^{-1}\rho_2 = (12\ldots n)$

Sett $\mu = \rho_2^{-1}\rho_1 \Rightarrow \mu(12\ldots n)\mu^{-1} = (12\ldots n)$

Vi har n skrivemåter for $(12\ldots n)$: $(2\ldots n1)$, $(3\ldots n12)$,

$\Rightarrow$ Anta n-sykler er $\frac{|S_n|}{n} = (n-1)!$

b) $\sigma \in S_n$

$C(\sigma) = \{\rho \in S_n \mid \rho\sigma = \sigma\rho\} = \{\rho \in S_n \mid \rho^{-1}\sigma\rho = \sigma\}$

S_n virker på S_n ved konjugasjon $\sigma \mapsto \rho^{-1}\sigma\rho$.

Gruppevirkning og $C(\sigma) = (S_n)_{\langle\sigma\rangle}$

isotropigruppa til σ

$$|S_n\sigma| = \frac{|S_n|}{|C(\sigma)|}$$

$$\Downarrow$$

$$|C(\sigma)| = \frac{|S_n|}{|S_n\sigma|} = \frac{n!}{(n-1)!} = n$$

↖ fra a)

Men $\sigma \in C(\sigma) \Rightarrow \langle\sigma\rangle \leq C(\sigma)$ og siden

$|\langle\sigma\rangle| = n = |C(\sigma)|$

følger det at $C(\sigma) = \langle\sigma\rangle$

2 $\left.\begin{array}{l}\sigma = (12345)\\ \tau = (15)(24)\end{array}\right\} \in S_5$

a) $\tau^2 = (15)(24)(15)(24) = id$

$\tau(3) = (15)(24)(3) = 3$

$\tau\sigma\tau = (15)(24)(12345)(15)(24)$

$\left.\begin{array}{l}1 \mapsto 1 \mapsto 5 \mapsto 1 \mapsto 1 \mapsto 5\\ 2 \mapsto 4 \mapsto 4 \mapsto 5 \mapsto 5 \mapsto 1\\ 3 \mapsto 3 \mapsto 3 \mapsto 4 \mapsto 2 \mapsto 2\\ 4 \mapsto 2 \mapsto 2 \mapsto 3 \mapsto 3 \mapsto 3\\ 5 \mapsto 5 \mapsto 1 \mapsto 2 \mapsto 4 \mapsto 4\end{array}\right\} (15432) = (12345)^{-1}$

$\tau\sigma\tau = \sigma^{-1} \Leftrightarrow \tau\sigma\tau\sigma = id$, dvs $(\tau\sigma)^2 = id$

$\tau\sigma = (15)(24)(12345) = (14)(23) \Rightarrow (\tau\sigma)^2 = id$

b) $N \subseteq S_5 \quad N = \{\rho \in S_5 \mid \rho \sigma \rho^{-1} = \sigma^{\varepsilon(\rho)}, \ \varepsilon(\rho) = \pm 1\}$

Undergruppe • $\rho_1, \rho_2 \in N$

$$\Rightarrow \quad \rho_1 \rho_2 \sigma (\rho_1 \rho_2)^{-1} = \rho_1 (\rho_2 \sigma \rho_2^{-1}) \rho_1^{-1}$$
$$= \rho_1 \sigma^{\varepsilon(\rho_2)} \rho_1^{-1}$$
$$= (\sigma^{\varepsilon(\rho_2)})^{\varepsilon(\rho_1)} = \sigma^{\varepsilon(\rho_2)\varepsilon(\rho_1)}$$

$\varepsilon(\rho_2) \cdot \varepsilon(\rho_1) = \pm 1 \Rightarrow \rho_1 \rho_2 \in N$

• $e \sigma e^{-1} = \sigma^1$ ok

• $\rho \in N \Rightarrow \rho \sigma \rho^{-1} = \sigma^{\varepsilon(\rho)}$

$$\Rightarrow \rho^{-1} \sigma^{\varepsilon(\rho)} \rho^{+1} = \sigma$$
$$\Rightarrow (\rho^{-1} \sigma^{\varepsilon(\rho)} \rho^{+1})^{\varepsilon(\rho)} = \sigma^{\varepsilon(\rho)}$$
$$\Rightarrow \rho^{-1} \sigma^{\varepsilon(\rho)\cdot\varepsilon(\rho)} \rho^{+1} = \rho^{-1} \sigma \rho = \sigma^{\varepsilon(\rho)} \text{ ok}$$
$\rho^{-1} \in N$

$\langle \sigma \rangle \leq N$ fordi $\sigma \sigma \sigma^{-1} = \sigma \in N$.

Normal: $\rho \in N \Rightarrow \rho \sigma \rho^{-1} = \sigma^{\varepsilon(\rho)} \in \langle \sigma \rangle$

c) $\varepsilon : N \longrightarrow \{\pm 1\}$

$\rho_1, \rho_2 \in N \Rightarrow (\rho_1 \rho_2) \sigma (\rho_1 \rho_2)^{-1} = \rho_1 \sigma^{\varepsilon(\rho_2)} \rho_1^{-1} = \sigma^{\varepsilon(\rho_2)\varepsilon(\rho_1)}$

$\Rightarrow \varepsilon(\rho_1 \rho_2) = \varepsilon(\rho_2)\varepsilon(\rho_1) = \varepsilon(\rho_1)\varepsilon(\rho_2)$ Homomorfi

Surjektiv fordi $\sigma = (12345)$ og $\rho = (15)(24)$ gir

$\rho \sigma \rho^{-1} = \sigma^{-1} \Rightarrow -1 \in \operatorname{Im} \varepsilon$

Dessuten $\sigma \sigma \sigma^{-1} = \sigma \Rightarrow 1 \in \operatorname{Im} \varepsilon$

$\operatorname{Ker}(\varepsilon) = \{\rho \in N \mid \rho \sigma \rho^{-1} = \sigma\} = C(\sigma)$

$$\Rightarrow |N| = |\{\pm 1\}| \cdot |\ker(\varepsilon)|$$
$$= 2 \cdot 5 = 10$$

3. $A, B \leq G$ normale abelske $\gcd(|A|, |B|) = 1$.

a) $x \in A \cap B \Rightarrow |x| \mid |A|$ og $|x| \mid |B| \Rightarrow |x| = 1$
$\Rightarrow x = e$.

$AB = \{xy \mid x \in A, y \in B\}$

$(x_1y_1)(x_2y_2)(x_1y_1)^{-1}(x_2y_2)^{-1}$

$= x_1 y_1 x_2 y_2 y_1^{-1} x_1^{-1} y_2^{-1} x_2^{-1}$

$= x_1 y_1 x_2 y_1^{-1} y_2 x_1^{-1} y_2^{-1} x_2^{-1} \in A$

$= x_1 y_1 x_1^{-1} x_1 x_2 y_2 y_1^{-1} x_1^{-1} x_2^{-1} x_2 y_2^{-1} x_2^{-1} \in B$

$\Rightarrow \in A \cap B$. Abelsk

$yx = yxy^{-1}y \in A \cdot B \Rightarrow |AB| = |A| \cdot |B|$

b) $A = \mathbb{Z}_n = \langle \alpha \rangle$, $B = \mathbb{Z}_n = \langle \beta \rangle$ $\Rightarrow AB = \langle \alpha\beta \rangle$

$(\alpha\beta)^s = \alpha\beta\alpha\beta \cdots \alpha\beta$

$= \alpha \cdot \beta\alpha\beta^{-1}\beta^2\alpha\beta^{-2}\beta^3\alpha\beta^{-3} \cdots \beta^{s-1}\alpha\beta^{-s+1}\beta^s$

$\in A$

$= e \Rightarrow s = k \cdot n$

Ved symmetri i A og B $\Rightarrow s = l \cdot m$

$\Rightarrow s = n \cdot m$

c) $|G| = 1001 = 7 \cdot 11 \cdot 13$

Antall Sylow 7-gruppe 1, 8, 15, 22, ... må dele 143
$\Rightarrow 1$ H_1

Antall Sylow 11-gruppe 1, 12, 23, ... må dele 91
$\Rightarrow 1$ H_2

Antall Sylow 13-grp 1, 14, 27, ... må dele 77
$\Rightarrow 1$ H_3

Alle er normale og sylowske

$G \simeq H_1 H_2 H_3$ sylowsk

4.

$K \leq L \ni i, a$

$i^2 = -1$

$a^2 = i$

a) $a^4 = (a^2)^2 = i^2 = -1$

$((1+i)a)^2 = (1+2i-1)a^2 = 2i\,i = -2$

$((1-i)a)^2 = (1-2i-1)a^2 = +2$

b)

$(x-a)(x+a)(x-ia)(x+ia)$

$= (x^2-a^2)(x^2-(ia)^2) = (x^2-a^2)(x^2+a^2) = x^4-a^4 = x^4+1$

c) x^4+1 irreducibelt / K

Reducibelt $\Leftrightarrow$ $a \in K$ (ella), $ia \in K$ (ella), $a(1+i) \in K$ (ella), $a(1-i) \in K$

ella $a^2 \in K$

$x^4+1 = (x^2-a^2)(x^2+a^2)$

$= (x^2-(a+ia)x+ia^2)(x^2+(a+ia)x+ia^2) = (x^2-a(1+i)x-1)(x^2+a(1+i)x-1)$

$= (x^2-(a-ia)x-ia^2)(x^2+(a-ia)x-ia^2) = (x^2-a(1-i)x+1)(x^2+a(1-i)x+1)$

Anta $i \notin K \Rightarrow a, ia \notin K$

Reducibelt $\Leftrightarrow$ $(a(1+i))^2 = -2$ (ella), $(a(1-i))^2 = 2$ (ella), $(a^2)^2 = -1$

er kvadrater i K.

Hvis $i \in K \Rightarrow$ Reducibelt $\Leftrightarrow$ $K = L$ eller -1 er kvadrat i K

$\Downarrow$

$2, -2, -1$ kvadrater

Hvis $a \in K \Rightarrow K = L \Rightarrow 2, -2, -1$ kvadrater

$\Rightarrow$ $2, -2, -1$ ikke kvadrater $\Leftrightarrow$ irreducibelt

d) $\mathbb{Z}_5$: $2^2 = -1$ $\quad$ $\mathbb{Z}_7$: $3^2 = 2$

$x^4+1 = (x^2-2)(x^2+2)$

$= x^4+2x^2-2x^2-4$

$= x^4+1$

$x^4+1 = (x^2-3x+1)(x^2+3x+1)$

$= x^4 + \cancel{3x^3} + x^2 - \cancel{3x^3} - \cancel{9x^2} - \cancel{3x} + \cancel{x^2} + \cancel{3x} + 1$

$= x^4+1$.