2017

1. $U(F) = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{pmatrix} \mid a,b,c \in F \right\}$

a) $\begin{pmatrix} 1 & 0 & 0 \\ a_1 & 1 & 0 \\ b_1 & c_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ a_2 & 1 & 0 \\ b_2 & c_2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ a_1+a_2 & 1 & 0 \\ * & c_1+c_2 & 1 \end{pmatrix}$ $\quad * = b_1 + c_1 a_2 + b_2$

/kha abdsh

$I_3 \in U(F)$

$\begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ * & -c & 1 \end{pmatrix}$ $\quad * = ac - b$

b) $H = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & a & 1 \end{pmatrix} \right\}$ $\quad a = c$ $\quad * = b_1 + a^2 + b_2$

abdsh

$$\begin{pmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ ac-b & -c & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ y & x & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ ac-b & -c & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ a+x & 1 & 0 \\ y+ax+b & x+c & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ y+(a-c)x & x & 1 \end{pmatrix} \in H$$

$H(\mathbb{Z}_2) = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & a & 0 \end{pmatrix} \right\} \simeq \mathbb{Z}_4 = \langle \alpha \rangle$

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}^3 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \quad ( \ )^4 = id$$

$\updownarrow$ $\quad \updownarrow$ $\quad \updownarrow$ $\quad \updownarrow$

$(1,0)$ $\quad (0,1)$ $\quad (1,1)$ $\quad (0,0)$

$\parallel$ $\quad \parallel$ $\quad \parallel$ $\quad \parallel$

$\alpha$ $\quad \alpha^2$ $\quad \alpha^3$ $\quad \alpha^4$

c) $U = U(\mathbb{Z}_2) \quad X \subseteq \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

$\parallel$

$\{(1,y,z) \mid y,z \in \mathbb{Z}_2\} = \{(1,0,0),(1,1,0),(1,0,1),(1,1,1)\}$

$$\begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{pmatrix} \begin{pmatrix} 1 \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ y+a \\ b+cy+z \end{pmatrix} \qquad \begin{pmatrix} 1 \\ y+a_1 \\ b_1+c_1y+z \end{pmatrix} = \begin{pmatrix} 1 \\ y+a_2 \\ b_2+c_2y+z \end{pmatrix}$$

$\Rightarrow a_1 = a_2 \quad b_1 + c_1 y = b_2 + c_2 y \quad \forall y = 0,1$

$\Rightarrow b_1 = b_2 \text{ og } c_1 = c_2.$

$\mathbb{Z}_4 \leq U(\mathbb{Z}_2)$

$$\left\{\begin{pmatrix}1&0&0\\a&1&0\\b&a&\phi\end{pmatrix}\right\} \hookrightarrow \left\{\begin{pmatrix}1&0&0\\a&1&0\\b&c&\phi\end{pmatrix}\right\} \longrightarrow \mathbb{Z}_2$$

$$(a,b,c) \longmapsto a-c$$

$\Rightarrow U(\mathbb{Z}_2) \supset D_4$ Unge permutasjon av orde 8, og ikke-abelsk

$$\begin{pmatrix}1&0&0\\1&1&0\\0&1&1\end{pmatrix}\begin{pmatrix}1\\x\\y\end{pmatrix} = \begin{pmatrix}1\\1+x\\x+y\end{pmatrix} \qquad \begin{array}{l}(0,0)\mapsto(1,0)\\(1,0)\mapsto(0,1)\\(0,1)\mapsto(1,1)\\(1,1)\mapsto(0,0)\end{array} \qquad \sigma = (1234)$$

$$\begin{pmatrix}1&0&0\\1&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1\\x\\y\end{pmatrix} = \begin{pmatrix}1\\1+x\\y\end{pmatrix} \qquad \begin{array}{l}(0,0)\mapsto(1,0)\\(1,0)\mapsto(0,0)\\(0,1)\mapsto(1,1)\\(1,1)\mapsto(0,1)\end{array} \qquad \mu = (12)(34)$$

14.52

$$U(\mathbb{Z}_2) = \langle \sigma, \mu \mid \sigma^4 = \mu^2 = e,\ \mu\sigma\mu = \sigma^{-1} \rangle$$

2. a) $H, K \leq G$ normal, $H \cap K = \{e\}$

$$H \times K \longrightarrow G$$
$$(h,k) \longmapsto hk$$

$$(h_1,k_1)(h_2,k_2) \mapsto h_1k_1h_2k_2 \Rightarrow$$
$$\| \qquad\qquad\qquad\qquad \Rightarrow$$
$$(h_1h_2, k_1k_2) \mapsto h_1h_2k_1k_2 \Rightarrow$$

$$\left.\begin{array}{l} hkh^{-1}k^{-1} = (hkh^{-1})k^{-1} \in K \\ \| \\ h(kh^{-1}k^{-1}) \in H \end{array}\right\} \Rightarrow hkh^{-1}k^{-1} = e \Leftrightarrow hk = kh.$$

$hk = e \Leftrightarrow k = h^{-1} \in H \cap K = \{e\}$ dus $(h,k) = (e,e)$

b) $|G| = 99 = 3^2 \cdot 11$

Sylow-11: 1, 12, 23, ... deler 99 $\Rightarrow$ Normal $H$

Sylow-3: 1, 4, 7, 10, 13, ... deler 99 $\Rightarrow$ Normal $K$

Begge er abelske $H \cap K = \{e\}$

$$\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_{11} \simeq \mathbb{Z}_3 \times \mathbb{Z}_{33}$$
$$\mathbb{Z}_9 \times \mathbb{Z}_{11} \simeq \mathbb{Z}_{99}$$

3.




$\omega = e^{\frac{2\pi i}{12}}$ $\quad f(x) = x^6 + 1 \in \mathbb{Q}[x]$

$f(\alpha) = 0 \Rightarrow f(\alpha^{2k+1}) = \alpha^{12k+6} + 1$

$\Downarrow$ $\qquad\qquad = (\alpha^6)^{2k+1} + 1$

$\alpha^6 = -1$ $\qquad\qquad = (-1)^{2k+1} + 1 = 0.$

a) $f(x) = x^6 + 1 = (x^2+1)(x^4 - x^2 + 1)$

b) $\mathbb{Q}(\omega)$ Zerfällkörper für $f(x)$

Nullst. $\omega, \underset{\| \; +i}{\omega^3}, \omega^5, -\omega, \underset{\| \; -i}{-\omega^3}, -\omega^5$

$\omega^3 = i$, $\quad \omega + \omega^{11} = e^{\frac{2\pi i}{12}} + e^{\frac{22\pi i}{12}} = \frac{\sqrt{3}}{2} + i\frac{1}{2} + \frac{\sqrt{3}}{2} - i\frac{1}{2} = \sqrt{3}$

$\Rightarrow \mathbb{Q}(i, \sqrt{3}) \subseteq \mathbb{Q}(\omega)$

$\omega = \frac{1}{2}\sqrt{3} + \frac{1}{2}i \Rightarrow \mathbb{Q}(\omega) \subseteq \mathbb{Q}(i, \sqrt{3})$.

$[\mathbb{Q}(\omega) : \mathbb{Q}] = [\mathbb{Q}(i,\sqrt{3}) : \mathbb{Q}(i)][\mathbb{Q}(i) : \mathbb{Q}] = 4$.

$G(\mathbb{Q}(\omega)/\mathbb{Q}(i)) \hookrightarrow G(\mathbb{Q}(\omega)/\mathbb{Q}) \to G(\mathbb{Q}(i)/\mathbb{Q})$

$\cong \mathbb{Z}_2$ $\quad \varphi \mapsto \varphi$ $\quad \psi$ $\quad \psi \longmapsto (i \mapsto -i) \cong \mathbb{Z}_2$

$\psi(\sqrt{3}) = \psi(\omega + \omega^{11}) = \psi(\omega) + \psi(\omega)^{11} = -\omega^5 + (-\omega^5)^{11} = -\omega^5 - \omega^7 = \omega^{11} + \omega = \sqrt{3}$

$\psi(\omega) = -\omega^5$

$\Rightarrow \psi(i) = \psi(\omega^3) = \psi(\omega)^3 = (-\omega^5)^3 = -\omega^{15} = -\omega^3 = -i$

$\psi^2(\omega) = \psi(\psi(\omega)) = \psi(-\omega^5)$

$= -\psi(\omega)^5 = -(-\omega^5)^5 = \omega$

$\varphi(\sqrt{3}) = -\sqrt{3}$

$\varphi(i) = i$ $\qquad G(\mathbb{Q}(\omega)/\mathbb{Q}) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$

4.

a) $F[x] \ni f(x)$ irreducible, $\deg(f) = n$

$K = F(f(x))$

$[K:F] = [K:F] = |G(K/F)|$

$n!$

$n$

$n$ odd $\quad \exists \delta \in K$ s.a $\delta^2 \in F$, $\delta \notin F$

$F \leq F(\delta) \leq K$

$\Rightarrow [K:F] = [K:F(\delta)] \cdot [F(\delta):F]$

$\text{irr}(\delta, F) = x^2 - \delta^2 \qquad 2$

$f(x)$ irredu $\Rightarrow x^2 - \delta^2 \nmid f(x)$

b) $\deg f = 3$

$f(\alpha_1) = f(\alpha_2) = f(\alpha_3) = 0$

$\alpha_1, \alpha_2, \alpha_3 \in K$

$S = \{\text{perm av } \alpha_1, \alpha_2, \alpha_3\}$

$G = G(K/\mathbb{Q}) \leq S$

$\delta = (\alpha_2 - \alpha_1)(\alpha_3 - \alpha_1)(\alpha_3 - \alpha_2) \in K$

$D = \delta^2$

$\sigma \in G$; $\sigma(D) = D$

$\Rightarrow D \in K_G = \mathbb{Q}$

$D$ ikke kvadrat i $\mathbb{Q} \Rightarrow \delta \notin \mathbb{Q}$

$[K:\mathbb{Q}] = [K:\mathbb{Q}(\delta)] \cdot [\mathbb{Q}(\delta):\mathbb{Q}]$

$6 \qquad\qquad 2$

2.3 ved a)

$|G(K/\mathbb{Q})| = 6$, $G(K/\mathbb{Q}) \leq S$

$\Rightarrow G(K/\mathbb{Q}) \cong S_3$

c) $\tau$, $\tau^2 = id$ (transposisjon) $\quad D = \delta^2$, $\delta \in \mathbb{Q}$

$\Rightarrow \tau(\delta) = \delta$

Anta $\tau = (12) \Rightarrow \tau(\delta) = \tau((\alpha_2 - \alpha_1)(\alpha_3 - \alpha_1)(\alpha_3 - \alpha_2))$

$= (\alpha_1 - \alpha_2)(\alpha_3 - \alpha_2)(\alpha_3 - \alpha_1) = -\delta$

Motsetning

$\Rightarrow \tau \notin G(K/\mathbb{Q})$

$$G(K/F(\delta)) \trianglelefteq G(K/F) \longrightarrow G(F(\delta)/F)$$
$$\cong$$
$$\mathbb{Z}_2$$

$\deg f = n$ $\alpha \in K$, $f(\alpha) = 0$

$[F(\alpha) : F] = n$ $\quad |G(F(\alpha)/F)| \mid |G(K/F)|$

$\Rightarrow |G(K/F)| = m \cdot n$

$\Rightarrow |G(K/F(\delta))| = m \cdot n \frac{1}{2} = \frac{m}{2} \cdot n$ siden $n$ er odde

$\Rightarrow m \in 2\mathbb{Z}$, $m = 2k$

$\Rightarrow |G(K/F)| = [K : F] = 2kn \geq 2n$.

# 2019

1 a) $\mathbb{Z}_{\geq 0}$ mangler additive inverse

b) $18 = 2 \cdot 3 \cdot 3$ Abelsk $\mathbb{Z}_{18}$, $\mathbb{Z}_3 \times \mathbb{Z}_6$.

c) $H = \langle (0,3) \rangle \leq \mathbb{Z}_2 \times \mathbb{Z}_9$

$H$, $(1,0)+H$, $(0,1)+H$, $(0,2)+H$, $(1,1)+H$, $(1,2)+H$

$$\frac{\mathbb{Z}_2 \times \mathbb{Z}_9}{\langle (0,3) \rangle} \simeq \mathbb{Z}_2 \times \mathbb{Z}_3 \simeq \mathbb{Z}_6$$

2. a) $\sigma = (26143)$
$\tau = (5213) \in S_6$

$$\sigma\tau = (26143)(5213) = (124356) \quad |\sigma\tau| = 6$$

b) $\sigma \in S_n$
$|\sigma| = 2k+1$ $\Rightarrow (\sigma^2)^m = \sigma^{2m} = id$

betyr $2k+1 \mid 2m$ dvs $2k+1 \mid m$.

$$(\sigma^2)^{2k+1} = (\sigma^{2k+1})^2 = id$$

c) $G \leq S_n$ slik at $\exists \sigma_2, \ldots, \sigma_n \in G$
med $\sigma_i(1) = i$ $2 \leq i \leq n$

$G$ virker transitivt på $\{1, \ldots, n\}$, dvs $\forall 1 \leq i,j \leq n \; \exists \sigma \in G$ slik at $\sigma(i) = j$

$$\sigma_j \sigma_i^{-1}(i) = \sigma_j(1) = j$$

3 a) $n \in \mathbb{Z}$ ikke prim $\Rightarrow$ $n\mathbb{Z} \subsetneq \mathbb{Z}$ ikke primideal

$\mathbb{Z}/n\mathbb{Z}$

$pq = n$ $\quad \bar{p}, \bar{q}$ $\quad \bar{p} \cdot \bar{q} = 0$.

b) $K = \frac{\mathbb{Z}_2[x]}{(x^3+x+1)}$ $\quad x^3+x+1$ irredusibel i $\mathbb{Z}_2$ fordi $1^3+1+1 \neq 0$, $0^3+0+1 \neq 0$

$I$

$[K : \mathbb{Z}_2] = 3 \Rightarrow |K| = 2^3 = 8$.

c) $p(x) \in \mathbb{Z}_2[x]$
$\deg(p(x)) \leq 2$.
$p(x) = ax^2+bx+c$

$$x^5 + I = x^2 x^3 + I = x^2(x^3+x+1-x-1)+I$$
$$= x^2(-x-1)+I = -x^3-x^2+I$$
$$= -x^3-x-1+x+1-x^2+I = -x^2+x+1+I$$

4. $K = \mathbb{Q}(x^3-2) \in \mathbb{Q}[x]$

a) $[K:\mathbb{Q}] = 6$ Basis $\{1, \alpha, \omega\alpha, \alpha^2, \omega\alpha^2, \omega\}$ $\alpha^3 = 2$ reell $\omega^3 = 1$ $\omega \neq 1$

$K = \mathbb{Q}(\alpha, \omega)$

$G(K/\mathbb{Q})$ 6 ingar elemnh av ordn 6

$\sigma: \alpha \mapsto \omega\alpha \Rightarrow \omega^2 \mapsto \frac{\omega^2\alpha}{\alpha} \mapsto \frac{\omega^2\alpha}{\omega\alpha} = \omega$
$\omega^2\alpha \mapsto \omega^3\alpha$

(12) dvs $\sigma: \begin{cases} \alpha \mapsto \omega\alpha \\ \omega \mapsto \omega^2 \end{cases}$

$\sigma^2 \begin{cases} \alpha \mapsto \omega\alpha \mapsto \omega^2\omega\alpha = \alpha \\ \omega \mapsto \omega^2 \mapsto \omega \end{cases}$ $\sigma^2 = id$

$\tau: \alpha \mapsto \omega\alpha$
$\omega\alpha \mapsto \omega^2\alpha \Rightarrow \omega = \frac{\omega\alpha}{\alpha} \mapsto \frac{\omega^2\alpha}{\omega\alpha} = \omega$

(123) $\tau: \begin{cases} \alpha \mapsto \omega\alpha \\ \omega \mapsto \omega \end{cases}$ $\tau^2: \begin{cases} \alpha \mapsto \omega^2\alpha \\ \omega \mapsto \omega \end{cases}$ $\tau^3 = id$

b) $k = \mathbb{Q}(f(x))$
$\deg(f) = 3$
irreducibel

$G(K/\mathbb{Q}) \simeq A_3 \Leftrightarrow K = \mathbb{Q}(\alpha)$, $f(\alpha) = 0$

$\Updownarrow$

$|G(K/\mathbb{Q})| = 3$

$\Updownarrow$

$[K:\mathbb{Q}] = 3 \Leftrightarrow [K:\mathbb{Q}(\alpha)] \cdot [\mathbb{Q}(\alpha):\mathbb{Q}] = 3$

$\Updownarrow$

$[K:\mathbb{Q}(\alpha)] = 1$

sidan $[\mathbb{Q}(\alpha):\mathbb{Q}] \neq 1$

c) $\deg(f) = 3 \Rightarrow \exists$ reell rot $\alpha$. $\Rightarrow \alpha \in K$.

dvs $\mathbb{Q}(\alpha) \leq K$

Hvis $\mathbb{Q}(\alpha) = K$, så $[K:\mathbb{Q}] = 3$, bas $\{1, \alpha, \alpha^2\}$

$\Rightarrow$ Alle røtt a reelle.