

§4 Groups

Def. a group is given by

- a set G
- a binary operation $* : G \times G \rightarrow G$
(i.e. $a * b$ makes sense as an elem. of G for $a, b \in G$)
- a distinguished element $e \in G$
- an operation $a \mapsto a^{-1}$ on G "inverse"
(i.e. a^{-1} makes sense as an elem. of G for $a \in G$)

such that :

- $*$ is associative : $(a * b) * c = a * (b * c)$
- e is an identity elem. for $*$: $e * a = a = a * e$
- $a * a^{-1} = e = a^{-1} * a$ $\square$

We often think of a group as a collection of reversible transforms of some system

e : doing nothing , a^{-1} : reversing the effect of a

Ex. $G = GL_n = \{ X \in M_n \mid \text{invertible matrix} \}$

$e = I_n$ identity mat. $X * Y$: matrix product.

$\leadsto I_n * X = X = X * I_n$, $X * X^{-1} = I_n = X^{-1} * X$, etc.

X represents an invertible linear transform on $\mathbb{R}^n$

$$v \mapsto Xv \quad (Xv)_i = \sum_{j=1}^n X_{ij} v_j$$

Ex. the n -th symmetric group S_n ($n=0,1,2,\dots$)

$S_n = \{ \sigma \mid \text{permutation of the numbers } 1, 2, \dots, n \}$

i.e. the collection of bijective maps from the set $\{1, \dots, n\}$ to itself

e : the identity map , $\sigma * \rho$: composition of maps

Some first properties of groups

Thm. 4.15 $a * b = a * c \Rightarrow b = c$ "cancellation"
 $b * a = \cancel{c * a} \Rightarrow b = c$

Thm. 4.16 given $a, b \in G$, the equations
 $a * x = b$, $y * a = b$ have unique solutions
for x, y

Thm. 4.17 e is the unique elem. satisfying
 $e * a = a = a * e$

given $a \in G$, a^{-1} is the unique elem. satisfying
 $a * a^{-1} = e = a^{-1} * a$

(unique sol. to $a * x = e = x * a$)

Cor. 4.18 $(a * b)^{-1} = b^{-1} * a^{-1}$

Rem 4.17 implies it's enough to specify $(G, *)$
Sketch

$$4.15: a^{-1} * (a * b) = (a^{-1} * a) * b = e * b = b$$

$$\text{gives } a * b = a * c \Rightarrow b = a^{-1} * (a * b) = a^{-1} * (a * c) = c$$

$$4.16: \begin{cases} x = a^{-1} * b \text{ will do by} & a * (a^{-1} * b) = (a * a^{-1}) * b \\ \text{existence:} & = e * b = b \end{cases}$$

uniqueness follows from 4.15

4.17: particular case of 4.16

4.18: compute $(a * b) * (b^{-1} * a^{-1})$ to get e

$$(a * b) * (b^{-1} * a^{-1}) = ((a * b) * b^{-1}) * a^{-1}$$

assoc.

$$\stackrel{\text{assoc.}}{=} (a * (b * b^{-1})) * a^{-1} = (a * e) * a^{-1} = a * a^{-1} = e$$

Important subclass: (or abelian) commutative groups

groups $(G, *)$ satisfying $a * b = b * a$

Ex. $(\mathbb{Z}, +)$, $(\mathbb{R}^n, +)$ are comm.

Non-ex. $(GL_2, \text{mat. prod.})$ is not comm.