

§ 13 Homomorphisms

We have one way to make sense of "comparison" of groups
isomorphism from $(G, *)$ to $(G', *')$: two cond.

- identification of two systems : one-to-one and onto map $\phi: G \rightarrow G'$
- compatibility with group structures : $\phi(a * b) = \phi(a) *' \phi(b)$

$\Rightarrow$ dropping the "identification" and only asking for
a map $\phi: G \rightarrow G'$ s.t. $\phi(a * b) = \phi(a) *' \phi(b)$

we get a more relaxed version of comparison

Def. a homomorphism from $(G, *)$ to $(G', *')$ is
such a map.

Examples

1. linear maps between vector spaces V, W

$$T: V \rightarrow W \text{ map s.t. } T(v+v') = T(v) + T(v') \\ \text{and } T(av) = aT(v)$$

2. $GL_n = \{n \times n \text{ invertible mats.}\} \rightarrow \{\text{nonzero numbers}\}$

$$X \mapsto \det(X)$$

$$\det(XY) = \det(X) \det(Y), \quad \det(X) \neq 0 \text{ for } \text{invertible } X$$

3. $S_n \rightarrow \mathbb{Z}_2 \quad \sigma \mapsto \begin{cases} 0 & \text{if } \sigma \text{ is even } (\tau_1 \dots \tau_k) \\ 1 & \text{if } \sigma \text{ is odd transp. } (\tau_1 \dots \tau_{k+1}) \end{cases}$

4. $\mathbb{Z}_2 \rightarrow \mathbb{Z}_5^\times = \{\text{integers up to congruence mod 5, remove multiples of 5}\}$, mult. as binary op.

$$\begin{array}{ccc} 0 \mapsto 1 & 1 \mapsto 4 (= -1) \\ \uparrow & \uparrow \\ \mathbb{Z}_2 & \mathbb{Z}_5^\times & \mathbb{Z}_2 & \mathbb{Z}_5^\times \end{array}$$

$$1+1=0 \text{ in } \mathbb{Z}_2 \quad \text{Corresp. to } 4^2=16=1 \text{ in } \mathbb{Z}_5^\times$$

Homomorphisms could be use to

- represent equations (1. $T(v) = w$ as eq. for v)
- extract important information (2., 3.)
- find subgroups with easier structure (4.)

Notation for maps $\phi: X \rightarrow Y$

- $A \subset X \rightsquigarrow$ image of A : $\phi[A] = \{ \phi(a) \mid a \in A \} \subset Y$
- image of ϕ (& range of ϕ) : $\phi[X]$
- $B \subset Y \rightsquigarrow$ inverse image (preimage) of B :
 $\phi^{-1}[B] = \{ x \in X \mid \phi(x) \in B \}$

(Writing $\phi(A)$, $\phi^{-1}(B)$ is very common)

$\hookrightarrow \text{Ker}$

Def $\phi: (G, *) \rightarrow (G', \cdot)$ group homomorphism

The kernel of ϕ is $\ker \phi = \{ g \in G \mid \phi(g) = e_{G'} \}$
 $= \phi^{-1}[\{ e_{G'} \}]$

Rem $\ker \phi \subset G$ is a subgroup:

$$\begin{array}{ccc} \phi(g) = e & , & \phi(h) = e \\ g \in \ker \phi & & h \in \ker \phi \end{array} \Rightarrow \underbrace{\phi(gh) = \phi(g)\phi(h) = e \cdot e = e}_{gh \in \ker \phi}$$

similarly $\phi(g^{-1}) = \phi(g)^{-1}$ implies $\ker \phi$
is closed under taking inverses

Prop. (cf. Thm 13.12)

G, G' : groups $\phi: G \rightarrow G'$: homomorphism

1. $H \subset G$ subgroup $\Rightarrow \phi[H] \subset G'$ subgroup
2. $K \subset G'$ subgroup $\Rightarrow \phi^{-1}[K] \subset G$ subgroup

Idea Combine $\phi(a)\phi(b) = \phi(ab)$, $\phi(a)^{-1} = \phi(a^{-1})$

with ~~$\phi(a), \phi(b) \in H$~~ $\phi(a), \phi(b) \in K$ (1.) or
 a, b (2.)

Thm 13.15 G, G' : groups, $G \xrightarrow{\phi} G'$: hom.

Put $H = \ker \phi$. Given $a \in G$, the subset

$$\{ b \in G \mid \phi(b) = \phi(a) \} = \phi^{-1}[\{ \phi(a) \}] \subset G$$

agrees with the left coset aH (also with Ha)

Proof Set $X = \{b \mid \phi(b) = \phi(a)\}$

Step 1 $aH \subset X$

An elem. in aH is of the form $ah, h \in H$
i.e. $\phi(h) = e_{G'}$

We want $\phi(ah) = \phi(a)$; use $\phi(ah) = \phi(a)\phi(h)$

Step 2 $X \subset aH$

Given b s.t. $\phi(b) = \phi(a)$, we want $h \in H$

s.t. $b = ah$; check $h = a^{-1}b$ works $\square$

Cor 13.18 $G \xrightarrow{\phi} G'$ hom is one-to-one if and
only if $\text{Ker } \phi = \{e\}$

Proof "Only if" from uniqueness of x s.t. $\phi(x) = e$
for one-to-one ϕ

"If" from $\text{Ker } \phi = \{e\} \Rightarrow$ ~~$\{b \mid \phi(b) = \phi(a)\}$~~ $\{b \mid \phi(b) = \phi(a)\}$ is
eq. to $\{a\}$

