

§ 14 Factor groups (quotient groups)

Idea : start with a group $(G, *)$

introduce a rule to identify elements of G

("ignore some information")

make the rule compatible with the binary op $*$

so that $G' = \{ \text{elements of } G, \text{ up to identification} \}$

becomes a group again

Example $(\mathbb{Z}_m, +)$ integers up to cong. mod. m
addition as group of

is obtained from $(\mathbb{Z}, +)$ by the identification

rule : $a \equiv b \pmod{m} \iff \exists k \in \mathbb{Z} : a - b = mk$

Q. How do we define such a rule?

1. use homomorphism $G \xrightarrow{\phi} G' \rightsquigarrow \phi(a) \stackrel{?}{=} \phi(b)$

2. use the coset rel $aH \stackrel{?}{=} bH$

Strategy 1

Given hom. $G \xrightarrow{\phi} G'$, consider the image $\phi[G]$

call this subgroup $\phi[G] \subset G'$ the factor group
of G with respect to ϕ

Good : once we know G' , it's a concrete construction

Bad : we need to know G' beforehand

Strategy 2

Let $H \subset G$ be a normal subgroup $\forall h \in H, g \in G : ghg^{-1} \in H$

Put $G/H = \{ [a] \mid a \in G, [a] = [b] \text{ when } a^{-1}b \in H \}$
"difference"

Prop / Def (14.4) G/H is a group with

- bin. op. $[a][b] = [ab]$

- identity $e_{G/H} = [e_G]$

- inverse $[a]^{-1} = [a]$

Proof idea : we need to check that these operations are well-defined ; if a and a' represent the same elem. in G/H , do we have $[a][b] = [a'][b]$?
 $[a]^{-1} = [a']^{-1}$?

Step 1 bin. op. is well-def'd

We want $[ab] = [a'b]$ from $[a] = [a']$

The assumption means $h = a^{-1}a'$ is in H

$$a' = a(a^{-1}a') = ah \Rightarrow a'b = ahb$$

We want $h' \in H$ s.t. $ahb = abh'$

$$ahb = ab b^{-1} h b \quad \text{use the normality to check} \\ h' = b^{-1} h b \text{ works}$$

Step 2 $e_{G/H} = [e_G]$ works as identity elem.

$$[e][a] = [ea] = [a], \text{ etc.}$$

Step 3 inverse op. is well-def'd

We want $[a^{-1}] = [a'^{-1}]$ from $[a] = [a']$

$$a' = ah \text{ for } h \in H \Rightarrow a'^{-1} = h^{-1}a^{-1}$$

Call G/H the factor group (or quotient group) of G by H $= a^{-1} a \underbrace{h^{-1}}_{\text{in } H} a^{-1}$
by normality, in H $\square$

Good : we can start from a normal subgroup $H \triangleleft G$
 $\rightarrow$ easier to construct new examples

Bad : G/H is defined in an indirect way

These two approaches look different, but they are essentially the same :

Thm 14.11 G, G' : groups $G \xrightarrow{\phi} G'$: homomorphism

$$\text{Set } H = \ker \phi = \{g \in G \mid \phi(g) = e_{H'}\}$$

Then G/H and $\phi[G]$ are isomorphic ($[a] \leftrightarrow \phi(a)$)

Proof We define an isom. $\mu : G/H \rightarrow \phi[G]$
by the above $\mu([a]) = \phi(a)$

We need to check :

- μ is a well-defined map
- μ is one-to-one and onto
- μ is compatible with group binary ops.

Step 1 well-def'd

$$[a] = [a'] \Rightarrow \mu(a) = \mu(a') \quad ? \quad \downarrow \text{this means } \phi(a) = \phi(a')$$

$$\uparrow \text{this means } a^{-1}a' \in H = \text{Ker } \phi \Leftrightarrow \phi(a^{-1}a') = e_H$$

$$\phi(a^{-1}a') = \phi(a)^{-1} \phi(a') \text{ since } \phi \text{ is a hom.}$$

$$\phi(a)^{-1} \phi(a') = e \Rightarrow \underbrace{\phi(a) \phi(a)^{-1} \phi(a')}_{\text{gives } \phi(a')} = \phi(a)$$

Step 2 compatibility with binary ops.

$$\underbrace{\mu([a][b])}_{\hookrightarrow \phi(ab)} \stackrel{?}{=} \underbrace{\mu([a])}_{\phi(a)} \underbrace{\mu([b])}_{\phi(b)} \quad (\mu \text{ is a hom.})$$

Step 3 μ is one-to-one

use Cor 13.18 (from 14. Feb.)

Step 4 μ is onto

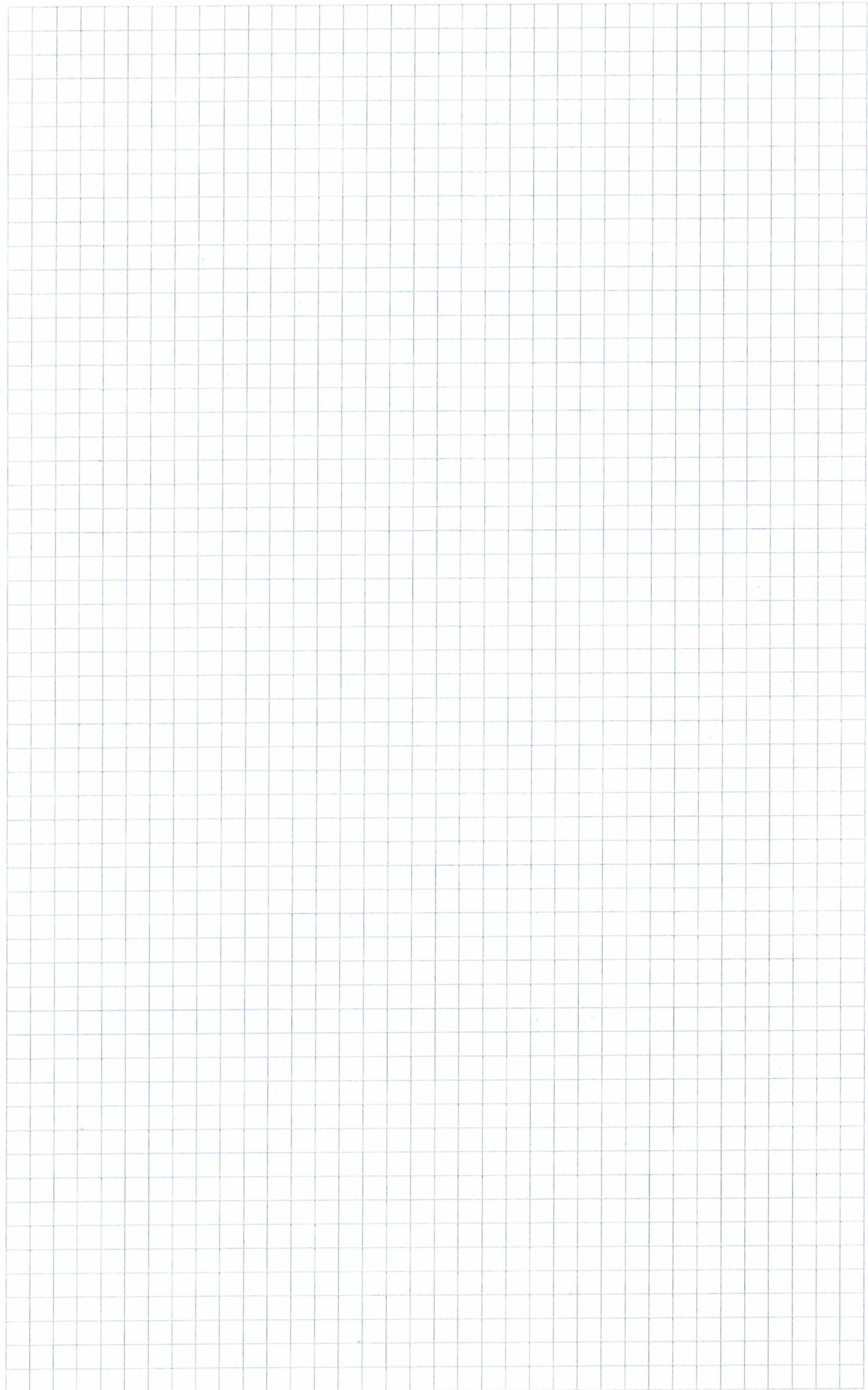
any elem of $\phi[G]$ is of the form $\mu([a])$ $\square$

We saw transformations of the form $\underbrace{x}_{\in G} \mapsto \underbrace{g x g^{-1}}_{\in G}$
several times

They are called inner automorphisms i_g (or Ad_g)

- automorphism : isomorphism from G to itself
- normal subgroup : subgroup $H \leq G$ s.t.

$$H = i_g[H] \text{ for all } g \in G$$



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