

§ 15 Simple groups, etc.

Def. a group G is simple if there are no normal subgroups other than $\{e\}$ and G

Examples

1. $\mathbb{Z}_p$ for prime p

Lagrange's thm implies $H \subset \mathbb{Z}_p$ subgroup

$$\Rightarrow |H| = 1 \text{ or } p$$

2. A_n for $n \geq 3$ or $n \geq 5$

Rem if G is a simple group, $G \xrightarrow{\phi} G'$ is a homomorphism, then ϕ is either one-to-one or trivial
 $\phi[G] = \{e_{G'}\}$

$\ker \phi$ is a normal subgroup of G , hence

$$\text{either } \begin{cases} \ker \phi = \{e_G\} \rightarrow \text{one-to-one} & (\text{Cor 13.18}) \\ \ker \phi = G \rightarrow \phi[G] = \{e_{G'}\} \end{cases}$$

Thm (15.18) $M \subset G$ normal subgroup

M is a maximal normal subgroup $\Leftrightarrow G/M$ is simple

Proof idea: (normal) subgroups of G/M (write K)

$$\Leftrightarrow \text{(normal) subgroups } H \subset G \text{ s.t. } H \supset M$$

given K , take $H = \{g \in G \mid [g] \in K\}$

given H , take $K = \{[g] \in G/M \mid g \in H\}$ $\square$

So we can "decompose" G by simple groups:

- find a maximal normal subgroup $M_1 \subset G$
 $\rightarrow$ get $K_1 = G/M_1$ simple
- then find a max. normal subgroup of M_1 ; write $M_2 \subset M_1$
 $\rightarrow$ get $K_2 = M_1/M_2$ simple
- ... continue until M_n is simple

Rem.

- this algorithm does not guarantee unique output

ex. $\mathbb{Z}_6 (\cong \mathbb{Z}_2 \times \mathbb{Z}_3)$ has max. normal subgroups

$$M_2 = \{0, 2, 4\} \quad \text{and} \quad M'_2 = \{0, 3\}$$

but if we forget the order of appearance

the coll. of simple groups $K_1, K_2, \dots$
is guaranteed to be the same (see §35)

- different groups can give the same output

$G \cong \mathbb{Z}_4$ and $G' = \mathbb{Z}_2 \times \mathbb{Z}_2$ both give

$$K_1 \cong \mathbb{Z}_2 \cong K_2$$

so how we combine $K_1, K_2, \dots$ to recover G
is a nontrivial task

Distinguished (normal) subgroups

Def. The center of G is $Z(G) = \{z \in G \mid \forall g \in G, gz = zg\}$

Ex. $G = GL_n = \{X \in M_n \mid \text{invertible}\}$

$$Z(G) = \{\alpha I_n \mid \alpha \neq 0 \text{ scalar}\}$$

Def. G group, $g, h \in G$

- the commutator of g and h is $ghg^{-1}h^{-1}$
(sometimes write $[g, h]$)

- the commutator subgroup (or derived subgroup)

is the subgroup of G generated by all commutators

(i.e. products $a_1 a_2 \dots$; $a_i = g_i h_i g_i^{-1} h_i^{-1}$)

(collect all)

Notation: $[G, G], D(G), G^{(1)}, G'$

Prop. $[G, G]$ is normal in G

Proof

Step 1 $a [g, h] a^{-1} = [a g a^{-1}, a h a^{-1}]$

use $a x a^{-1} a y a^{-1} = a x y a^{-1}, (a x a^{-1})^{-1} = a x^{-1} a^{-1}$

Step 2 the claim $[G, G]$ normal

given $a_i = g_i h_i g_i^{-1} h_i^{-1} = [g_i, h_i]$

$$a (a_1 a_2 \dots a_n) a^{-1} \stackrel{\text{step 1}}{=} a'_1 a'_2 \dots a'_n$$

with $a'_i = [a g_i a^{-1}, a h_i a^{-1}]$

□

Why consider $[G, G]$?

- In the quotient group $G/[G, G]$ we have

$$[a] = e \quad \text{for any } a = [g, h]$$

- $a = [g, h]$ satisfies $a h g = g h$

$$\leadsto \underbrace{[a]}_e [h] [g] = [g] [h] \quad \text{in } G/[G, G]$$

$\leadsto G/[G, G] \stackrel{e}{\text{is a commutative group}}$

(Abelianization / maximal comm. quotient of G)

Handwritten header text at the top of the page.

First paragraph of handwritten text.

Second paragraph of handwritten text.

Third paragraph of handwritten text.

Fourth paragraph of handwritten text.

Fifth paragraph of handwritten text.

Sixth paragraph of handwritten text.

Seventh paragraph of handwritten text.

Eighth paragraph of handwritten text.

Ninth paragraph of handwritten text.

Tenth paragraph of handwritten text.

Eleventh paragraph of handwritten text.

Twelfth paragraph of handwritten text.