

§ 16 Group action on a set

We often think of groups as collections of invertible transformations on some system

- rotations / flips preserving triangle, rectangle, square, ...
- invertible linear transforms on $\mathbb{R}^n$, ...

→ enough to make sense of $\overbrace{gx}^{\text{another "thing"}}$
to capture this $\uparrow \nwarrow$ "thing in the system"
elem. in the group

Def X : set G : group

an action of G on X is given by a map $G \times X \rightarrow X$
such that (write gx for its value)

- $ex = x$ for all $x \in X$
- $\underbrace{(gh)}_{\text{group bin op}} x = g(\underbrace{hx}_{\text{the effect of } G \times X \rightarrow X})$ for all $g, h \in G, x \in X$

We also say X is a G -set, or write $G \curvearrowright X$

Rem. (cf. Thm 16.3) recall $S_X = \{ \sigma : X \rightarrow X \mid \text{one-to-one, onto} \}$
action of G on $X \cong$ homomorphism $G \xrightarrow{\phi} S_X$

$$gx \leftrightarrow \phi(g)(x) \quad \text{i.e. } \sigma(x) \text{ for } \sigma = \phi(g)$$

Thm / Def $G \curvearrowright X$: action, $x \in X$

The subset $G_x = \{ g \in G \mid gx = x \} \subset G$ is a subgroup

Call this the isotropy subgroup of x (or stabilizer subgroup)
 $\text{Stab}_G(x)$

Proof. We want:

- $e \in G_x$
 - $g, h \in G_x \Rightarrow gh \in G_x$
 - $g \in G_x \Rightarrow g^{-1} \in G_x$
- follow from def. of action

→ ~~verify~~, compute $g^{-1}gx$ in two ways

cont

$$(g^{-1}g)x = ex = x$$

$$g^{-1}(gx) = g^{-1}x$$

Def. $G \curvearrowright X$, $x \in X$

$Gx = \{gx \mid g \in G\}$ is called the orbit of x under G
 $\uparrow$ not to be confused with G_x

Rem (cf. Thm 16.14) given $x, y \in X$

either $Gx = Gy$ or $Gx \cap Gy = \emptyset$

(equivalently: $y \in Gx \Rightarrow Gx = Gy$)

$$y = gx \Rightarrow \begin{cases} hx = hg^{-1}gx = hg^{-1}y \in Gy \\ \text{elem. of } Gx \\ ky = kgx \in Gx \\ \text{elem. of } Gy \end{cases}$$

Thm 16.16 $G \curvearrowright X$, $x \in X$

There is a one-to-one correspondence between

- the left cosets of G_x in G ; $hG_x \subset G$
- the elements of Gx

given by $hG_x \leftrightarrow hx$

Conseq. $|Gx| = (G : G_x)$ index of G_x in G
(we wrote $[G : G_x]$ before)

$$|G| < \infty \Rightarrow |Gx| \text{ divides } |G|$$

Proof of Thm

Write $G/G_x = \{hG_x \mid h \in G\}$ (ignore repeats)

Want: $\mu: G/G_x \rightarrow Gx$, $hG_x \mapsto hx$ is

- well-defined map
- one-to-one
- onto

Step 1 well-def'd

suppose $h G_x = k G_x$; want $hx = kx$

$h \in h G_x = k G_x \Rightarrow h = kg$ for $g \in G_x$

Then $hx = (kg)x = k(gx) = kx$

Step 2 one-to-one

suppose $hx = kx$; want $h G_x = k G_x$

$g = h^{-1}k$ satisfies $\begin{cases} gx = x & (\text{compute } h^{-1}(hx)) \\ hg = k & \Rightarrow k \in h G_x \end{cases}$

Step 3 onto ; obvious from const. $\square$

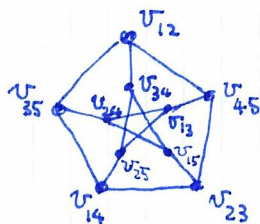
Ex. $SO(3) \curvearrowright S^2$
 3×3 -rot. mat. 2-dim sphere

$x \in S^2 \rightarrow$ stabilizer grp. $\cong SO(2)$

$$x = (1, 0, 0) \curvearrowright \text{Stab}_{SO(3)}(x) = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{bmatrix} \mid \theta \in \mathbb{R} \right\}$$

Ex. $S_5 \curvearrowright$ ^{vertices of} Petersen graph

$6v_{ab}$ $\uparrow$ $\begin{cases} \text{vertices } v_{ab} & a, b = 1, \dots, 5, a \neq b \text{ unordered pair} \\ \text{edge between } v_{ab} \text{ and } v_{cd} & \text{if } a, b, c, d \text{ are all different} \end{cases}$
 $= v_{\sigma(a), \sigma(b)}$



$$\text{Stab}_{S_5}(v_{12}) = \{ gh \mid g \in S_2, h \in S_{\{3,4,5\}} \} \cong S_2 \times S_3$$

1. The first part of the report discusses the background of the project and the objectives of the study. It also outlines the methodology used for data collection and analysis.

2. The second part of the report presents the results of the study, including the findings of the surveys and interviews. It also discusses the implications of the findings for the project and the organization.

3. The third part of the report provides a conclusion and recommendations for future research and action. It also includes a list of references and an appendix of additional data.

4. The fourth part of the report contains a detailed analysis of the data, including a comparison of the results with previous studies and a discussion of the limitations of the study. It also includes a list of references and an appendix of additional data.

5. The fifth part of the report provides a summary of the findings and a final conclusion. It also includes a list of references and an appendix of additional data.

6. The sixth part of the report contains a detailed analysis of the data, including a comparison of the results with previous studies and a discussion of the limitations of the study. It also includes a list of references and an appendix of additional data.

7. The seventh part of the report provides a summary of the findings and a final conclusion. It also includes a list of references and an appendix of additional data.

8. The eighth part of the report contains a detailed analysis of the data, including a comparison of the results with previous studies and a discussion of the limitations of the study. It also includes a list of references and an appendix of additional data.

9. The ninth part of the report provides a summary of the findings and a final conclusion. It also includes a list of references and an appendix of additional data.

10. The tenth part of the report contains a detailed analysis of the data, including a comparison of the results with previous studies and a discussion of the limitations of the study. It also includes a list of references and an appendix of additional data.