

§21 Field of quotients for integral domain

integral domains : rings like $\mathbb{Z}$

fields : rings like $\mathbb{Q}$

~~the~~ the elements of $\mathbb{Q}$ are fractions $\frac{a}{b}$
 a, b integers, $b \neq 0$

$\rightarrow$ can we do something like this to integ. doms.
 in general?

Obs. addition & multiplication in $\mathbb{Q}$

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + cb}{bd}, \quad \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

these formulas only use ring operations

We need to sort out:

- when two fractions represent the same number

$$\text{e.g. } \frac{2}{3} = \frac{4}{6} = \frac{-6}{-9} = \dots$$

- how to get consistency / well-definedness of
 ring operations for $\frac{a}{b}, \frac{c}{d}$ $a, b, c, d \in R$
 when R is an integral dom.

Why should we start from integral domains?

if we start from $\mathbb{Z}_6$ and consider $\frac{a}{b}$ for $b \neq 0$

$$\frac{1}{2} + \frac{1}{3} = \frac{5}{6} = \frac{5}{0}, \quad \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6} = \frac{1}{0}, \dots$$

i.e. we cannot have zero-divisors as denominators

How to make sense of the system of fractions

R : integral domain (commutative, $\exists 1$, $ab=0 \Rightarrow a=0$ or $b=0$)

- consider $\frac{a}{b}$, $b \neq 0$ (formally: ordered pair (a, b)
 s.t. $b \neq 0$)

- impose relation $\frac{a}{b} = \frac{c}{d}$ when $ad = bc$

(cont.) formally : consider equiv. rel. $\sim$ defined as
 $(a, b) \sim (c, d)$ when $ad = bc$
and take the equiv. classes

- define the addition bin. op. as

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + cb}{bd} \quad ; \text{ check that this is well-defined}$$

- define the mult. bin. op. as

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd} \quad ; \text{ check that this is well-defined}$$

- check that the above ops. define a ring

- addition def's. a comm. grp.

- mult. is associative

- distributive law holds $\left(\frac{a_1}{b_1} + \frac{a_2}{b_2}\right) \frac{a_3}{b_3} = \frac{a_1 a_3}{b_1 b_3} + \frac{a_2 a_3}{b_2 b_3}$

- check that this ring is a field

- multiplicative identity $\frac{1}{1}$ ($= \frac{a}{a}$ for $a \neq 0$)

- commutativity $\frac{a}{b} \cdot \frac{c}{d} = \frac{c}{d} \cdot \frac{a}{b}$

- inverse for nonzero elems. $\frac{b}{a}$;

check that this is well-defined

$\Rightarrow$ we call this field the field of quotients
or field of fractions of R

Ex. $R = \{ a_0 + a_1 X + \dots + a_k X^k \mid a_i \in \mathbb{R} \}$
ring of polynomials

$\Rightarrow$ field of fractions $= \{ \frac{p(X)}{q(X)} \mid q(X) \neq 0 \}$

field of rational functions

§22 Rings of polynomials

We want to treat a ring R as a system of "coefficients" for algebraic equations

$$(\text{like } a_0 + a_1x + \dots + a_kx^k = b)$$

A polynomial with coefficients in R

$$a_0 + a_1X + \dots + a_kX^k \quad (a_i \in R)$$

X : indeterminate / variable

$\leadsto$ we can make sense of addition & multiplication of polynomials in the usual way

$\leadsto R[X] = \{ \text{polynoms. with coeffs. in } R \}$
becomes a ring

Formally speaking, it's enough to remember the coefficients $a_0, \dots, a_k$ to recover the polynomial. $a_0 + a_1X + \dots + a_kX^k$

$\leadsto$ sequence $(a_n)_{n=0}^{\infty}$ s.t. $a_n = 0$ for $n > k$

$R[X]$ can be modeled as $\{ (a_n)_{n=0}^{\infty} \mid a_n \in R$
 $\exists k : a_n = 0 \text{ for } n > k \}$

similarly: if we have $X_1, \dots, X_m$

$R[X_1, \dots, X_m] = \{ a_{(0, \dots, 0)} + a_{(1, \dots, 0)}X_1 + a_{(0, 1, \dots, 0)}X_2 + \dots \}$
index $(d_1, \dots, d_m)$ $d_i \in \mathbb{N}$,
 $a_d = 0$ except for finitely many $d = (d_1, \dots, d_m)$

1. The first part of the paper is devoted to a general discussion of the problem.

2. In the second part we shall consider the case of a homogeneous medium.

3. The third part is devoted to the study of the properties of the solutions.

4. In the fourth part we shall consider the case of an inhomogeneous medium.

5. The fifth part is devoted to the study of the properties of the solutions.

6. In the sixth part we shall consider the case of a homogeneous medium.

7. The seventh part is devoted to the study of the properties of the solutions.

8. In the eighth part we shall consider the case of an inhomogeneous medium.

9. The ninth part is devoted to the study of the properties of the solutions.

10. In the tenth part we shall consider the case of a homogeneous medium.

11. The eleventh part is devoted to the study of the properties of the solutions.

12. In the twelfth part we shall consider the case of an inhomogeneous medium.

13. The thirteenth part is devoted to the study of the properties of the solutions.