

§27 Prime and maximal ideals

Today's convention: R commutative ring with multiplicative identity

ideal $N \subset R$ $\begin{cases} a \in N, b \in R \Rightarrow ab \in N \\ a \in N, b \in N \Rightarrow a+b \in N, \text{ etc.} \end{cases}$

$\Rightarrow$ quotient ring $R/N = \{[a] \mid a \in R, [a] = [b] \text{ when } a-b \in N\}$

When is R/N

- an integral domain? i.e. $[a][b] = 0 \Rightarrow [a] = 0$ or $[b] = 0$

- a field? i.e. $[a] \neq 0 \Rightarrow \exists b \text{ s.t. } [a][b] = 1$

Ex. $R = \mathbb{Z} \rightarrow$ ideals are $n\mathbb{Z} = \{nk \mid k \in \mathbb{Z}\}$

$\rightarrow$ quotients are $\mathbb{Z}_n = \{\text{int. up to cong. mod } n\}$

this is an integ. dom. if and only if n is prime

(if $n \neq 0$, then $\mathbb{Z}_n$ is also a field), $\mathbb{Z}/0\mathbb{Z} \cong \mathbb{Z}$

Ex. $R = \mathbb{Z}[X]$, $N = \{a_1X + \dots + a_nX^n \mid k=1,2,\dots, a_i \in \mathbb{Z}\}$

$\mathbb{Z}[X]/N \cong \mathbb{Z}$ (take the const. term)

int. dom., not field

Def 27.7 maximal ideal of R : an ideal $N \subset R$ s.t.

- $N \neq R$

- there is no ideal N' s.t. $N \subsetneq N' \subsetneq R$

Thm 27.9 Suppose $N \subset R$ is an ideal

Then N is a maximal ideal of $R \Leftrightarrow R/N$ is a field

Proof for " $\Rightarrow$ ": we want $[a] \neq 0 \Rightarrow \exists b \text{ s.t. } [a][b] = [1]$ in R/N

Step 1 R' comm. ring with mult. id., $a' \in R'$

if a' does not have a multiplic. inv., then

$N' = \{a'b' \mid b' \in R'\}$ is a proper ideal of R'

$\therefore 1_{R'}$ is not in N' by assumption on a'

N' is an ideal from $(a'b')c' = a'(b'c')$
 $\forall a'b' + a'c' = a'(b'+c')$

Step 2 $[a] \neq 0$ in R/M , M max. ideal $\Rightarrow \exists b : [a][b] = 1$

Proof by contradiction: if there is no such b

Step 1 gives $N' = \{[a][b] \mid b \in R\} \subset R/M$
 proper ideal

Take $M' = \{c \in R \mid [c] \in N'\} = aR + M$

this is an ideal of R , satisfies $M \subsetneq M' \subsetneq R$

This contradicts the maximality of M

Proof for " $\Leftarrow$ ": we want $M \subsetneq M' \subsetneq R \Rightarrow M' = R$
 ideal

Step 1 if $a \notin M$ then $\{ab + c \mid b \in R, c \in M\} = R$
 $aR + M$

$\therefore [a] \neq 0$ in $R/M \leftarrow$ field

$\leadsto \exists b$ s.t. $[a][b] = 1$ i.e. $ab - 1 \in M$

Given $d \in R$ $(ab - 1)d \in M$ means

$d = abd + c$ for $c = (ab - 1)d$

Step 2 $M' = R$

Take $a \in M' \setminus M$ Then $aR + M \subset M'$

Step 1 forces $M' = R$ $\square$

Thm R has a maximal ideal

Sketch: consider the collection $\{N \subset R \mid \text{ideal}, 1 \notin N\}$

take an increasing seq. $N_1 \subset N_2 \subset \dots$ in this coll.

Set $N_\infty = \bigcup_{k=1}^{\infty} N_k$ ("the limit of N_k ")

N_∞ is an ideal, $1 \notin N_\infty$

If N_∞ is not maximal, continue taking $N_1' = N_\infty \subset N_2' \subset \dots$
 and repeat this argument

(formally, use Zorn's lemma from set theory)

Ex. $R = \{ (a_n)_{n=1}^{\infty} \mid a_n \in \mathbb{R}, \text{ bounded} \}$

$$N = \{ (a_n)_{n=1}^{\infty} \mid a_n \xrightarrow{n \rightarrow \infty} 0 \}$$

$\leadsto$ maximal ideals of R/N corresp. to
"limit schemes" for bounded sequences

Def. 27.13 prime ideal of R : an ideal $N \subset R$
such that $ab \in N \Rightarrow a \in N$ or $b \in N$ holds

Thm 27.15 Suppose $N \subset R$ is an ideal

Then N is a prime ideal of R

$\Leftrightarrow R/N$ is an integral domain

Cor. $N \subset R$ maximal ideal $\Rightarrow$ prime ideal

1. The first part of the paper is devoted to the study of the

properties of the function $f(x)$ defined by the equation

$$f(x) = \int_0^x \frac{1}{1+t^2} dt$$

It is well known that this function is increasing and concave down.

The second part of the paper is devoted to the study of the

properties of the function $g(x)$ defined by the equation

$$g(x) = \int_0^x \frac{1}{1+t^2} dt$$

It is well known that this function is increasing and concave down.

The third part of the paper is devoted to the study of the

properties of the function $h(x)$ defined by the equation

1. The first part of the paper is devoted to the study of the

properties of the function

2. The second part of the paper is devoted to the study of the

properties of the function