

§27 continued

Recall: if F is a field, Euclid division algorithm

$$\text{gives } f(X), g(X) \in F[X] \Rightarrow f(X) = g(X)q(X) + r(X) \\ \text{with } r(X) = 0 \text{ or } \deg r(X) < \deg g(X)$$

What does it say about ideals of $F[X]$?

Def. R : comm. ring with mult. identity

Given $a \in R$ the set $\{ab \mid b \in R\}$ is called
the principal ideal generated by a
We write $\langle a \rangle$ or aR

Caution: we used the same notation for cyclic
subgroups in the setting of groups.

in general $\langle x \rangle =$ a system generated by x
 $\uparrow$ depends on the context

Def. if any ideal of R is of the form aR
we say R is a principal ideal domain

Ex. $\mathbb{Z}$ is a PID: ideals = $n\mathbb{Z}$ for some n

Thm 27.24 F : field

$F[X]$ is a PID, i.e. any ideal of $F[X]$
is of the form $\{g(X) \cdot h(X) \mid h(X) \in F[X]\}$
for some fixed $g(X) \in F[X]$

Proof Take an ideal $N \subset F[X]$

Idea: if $N = g(X)F[X]$, then $g(X)$ has
the minimal degree among the nonzero elements of N

Take $g(X) \in N$ as such, so $f(X) \in N \Rightarrow \deg g(X) \leq \deg f(X)$

We want $f(X) = g(X)q(X)$ from $f(X) \in N$

