

April Fool's day edition : modules over rings

Def. R : ringa module over R (or left R -module) is
(left) given by

- commutative group M
- binary op. $R \times M \rightarrow M$ (write $a \cdot x$ $\begin{smallmatrix} a \in R \\ x \in M \end{smallmatrix}$)

s.t. $(a+b)x = ax + bx$, $a(x+y) = ax + ay$
 $(ab)x = a(bx)$ hold.

Motivation } if $1 \in R$ exists we assume $1x = x$

- if R is a field (e.g. $\mathbb{R}, \mathbb{C}$) this is vector space over R
- commutative groups $\equiv \mathbb{Z}$ -modules
(interpret $n \cdot x$ as $\underbrace{x + \dots + x}_{n \text{ times}}$)
- for comm. ring R : ideals $\equiv R$ -submodules of R
- ring of differential operators ~~$R = \{ \sum_{k=0}^N f_k(x) \frac{d}{dx} \}$~~
 $R = \{ \sum_{k=0}^N f_k(x) \left(\frac{d}{dx} \right)^k \mid N=0,1,\dots, f_k(x) \text{ polynom} \}$
 or multivariable variants

$\leadsto M = \{ f(x) \mid \text{differentiable} \}$ becomes
 left R -module

Recall on PID

R is a PID if $\boxed{ab=0 \Rightarrow a=0 \text{ or } b=0}$
 comm. ring with mult. id. of the form $\{ab \mid b \in R\}$

$a \in R$ is irreducible if $a = bc$ implies b unit
 or c unit

- any elem $a \in R$ has prod. decomp $a = p_1 \dots p_n$

cont. p_i irreducible

- given $a, b \in R$ "the greatest common divisor".
 $c = \gcd(a, b)$ can be defined by $cR = aR \oplus bR$
- "the least common multiple" by $cR = aR \cap bR$

Annihilators : solution to $ax = 0$

Given a left R -module M ; R comm.

- $\text{Ann}(M) = \{a \in R \mid ax = 0 \text{ for all } x \in M\} \subset R$
(ideal of R)
- $\text{Ann}_M(a) = \{x \in M \mid ax = 0\} \subset M$
submodule
- For irred. $p \in R$; $M_p = \bigcup_{k=1}^{\infty} \text{Ann}_M(p^k)$
 p -primary component of M

Prop. $a, b \in R \xrightarrow{\text{PID}} ab \in \text{Ann}(M)$ and $\gcd(a, b) = 1$

$$\leadsto M = \text{Ann}_M(a) \oplus \text{Ann}_M(b)$$

i.e. any $x \in M$ can be written as $x = y + z$
sit. $ay = 0, bz = 0$ in a unique way

Proof. Existence of decomp: write $1 \in 1 \cdot R = aR + bR$

$$\text{as } 1 = as + br$$

$$\leadsto x = 1x = \underbrace{asx}_z + \underbrace{brx}_y \quad \text{will do}$$

Uniqueness. Enough to check $\text{Ann}_M(a) \cap \text{Ann}_M(b) = 0$

again use $x = \underbrace{asx}_0 + brx = 0$ by $x \in \text{Ann}_M(b)$
"0 by $x \in \text{Ann}_M(a)$ "

Def. a left R -module M is a torsion module

if : for each $x \in M$ there is nonzero $a \in R$
sit. $ax = 0$

Prop. M finitely generated torsion module over R ^{PID}

Then $\exists a \neq 0 \in R$ s.t. $\text{Ann}(M) = aR$

$\therefore$ Take a generating set $\{x_1, \dots, x_k\} \subset M$

Find $a_i \neq 0$ s.t. $a_i x_i = 0$

$\leadsto b = a_1 \dots a_k$ is a nonzero elem in $\text{Ann}(M)$

$\leadsto \text{Ann}(M)$ is a nonzero ideal of R $\square$

Cor M fin. gen. tors. mod over R

$\leadsto M \cong M_{p_1} \oplus \dots \oplus M_{p_k}$ $p_1, \dots, p_k$ irred. elem of R

(primary decomposition of M)

Th'm M fin. gen. tors. mod. over R (PID)

Then $M \cong R/a_1R \oplus \dots \oplus R/a_nR$

for nonzero elements $a_1, \dots, a_n \in R$

Cor. finite comm. groups are isom. to

$$\mathbb{Z}/m_1\mathbb{Z} \oplus \dots \oplus \mathbb{Z}/m_n\mathbb{Z}$$

for some pos. integers m_i

Proof of Th'm We may assume $M = M_p$

for some irred. $p \in R$ so $p^k x = 0$ for all $x \in M$
(with big k)

We argue by induction on the num. of generators

Suppose M is generated by $x_1, \dots, x_n$

Choose k : $p^k x_n = 0$ and $p^{k-1} x_n \neq 0$

We may assume $p^k x_i = 0$ for all i

Set $N = M/Rx_n$ ~~submod~~ ^{img of} generated by $x_1, \dots, x_{n-1}$

By induction hyp. $N \cong R/p^k R \oplus \dots \oplus R/p^{k_{n-1}} R$

If we can find $N \hookrightarrow M$ R -mod hom

s.t. $\phi(\sigma(v)) = v$ for $v \in N$

$\uparrow$ quot. map

then $M \cong N \oplus R/p^k R$

Take $y_i \in M$ s.t. $\phi(y_i) =$ ^{ing of 1 in the} i -th factor $R/p^k R$

$$\tilde{x}_i = y_i - s_i \otimes x_N \quad s_i \text{ def'd by}$$

$$p^{k_i} y_i = r_i x_N, \quad r_i = p^{k_i} s_i$$

Then $N \rightarrow M, \quad \phi(y_i) \mapsto \tilde{x}_i$ is well defined &

$$(p^{k_i} \tilde{x}_i = p^{k_i} y_i - \underbrace{p^{k_i} s_i}_{r_i} x_N = 0)$$