

§31 Algebraic extensions

Recall : $E \supset F$ extension (inclusion) of fields
 $\alpha \in E$ is algebraic over F if there is some
 nonzero polynom. $f(X) \in F[X]$ s.t. $f(\alpha) = 0$

Def. $E \supset F$ is an algebraic extension if
 any element $\alpha \in E$ is algebraic over F

Example $\mathbb{C} \supset \mathbb{R}$ is algebraic; given $\alpha \in \mathbb{C}$

- if $\alpha \in \mathbb{R}$ then $f(X) = X - \alpha$ will do
- if $\alpha \notin \mathbb{R}$ $f(X) = (X - \alpha)(X - \bar{\alpha}) = X^2 - 2\operatorname{Re}(\alpha)X + |\alpha|^2$ will do

Non-example $\mathbb{C} \supset \mathbb{Q}$ is not algebraic;

$\pi, e \in \mathbb{C}$ are not algebraic over $\mathbb{Q}$

Important class of alg. exts : finite extensions

Def $E \supset F$ is finite if $\dim_F E$ is finite

i.e. $\exists \alpha_1, \dots, \alpha_n \in E$ that becomes a basis over F

(any $\beta \in E$ can be written as $\beta = \sum_{i=1}^n \alpha_i \gamma_i$ ~~⊗~~
 with $\gamma_i \in F$ in a unique way)

Rem.) Write $[E:F] = \dim_F E$, call it the degree of E over F

Thm 31.3 If $E \supset F$ is a finite extension, then it is
 algebraic; i.e. any $\alpha \in E$ is a root of some $f \in F[X]$
nonzero

Proof. Set $n = [E:F]$

Consider the $n+1$ elements $1, \alpha, \dots, \alpha^n$

They cannot be linearly independent over F

$\Rightarrow \exists \beta_0, \dots, \beta_n \in F$ not all zero s.t. $\beta_0 \cdot 1 + \beta_1 \alpha + \dots + \beta_n \alpha^n = 0$

This gives a desired polynom. $f(X) = \sum_{i=0}^n \beta_i X^i$

Idea: we get alg. ext. $F(\alpha_1, \dots, \alpha_n) \supset F$
 from adding roots α_i for polynoms $f_i(X) \in F[X]$
 $\rightarrow$ collect all such extensions and take union

How to implement this

Step 1 for each nonconst $f(X) \in F[X]$

find a comm. F -alg. A_f s.t.

$$f(X) = \gamma (X - \alpha_1) \cdots (X - \alpha_k) \quad k = \deg f(X)$$

holds with some $\alpha_i, \gamma \in A_f$

How: write $f(X) = c_0 + c_1 X + \cdots + c_k X^k$

$$A_f = K[y_1, \dots, y_k \mid e_j(y_1, \dots, y_k) = \frac{(-1)^{k-j} c_j}{c_k}]$$

j-th elem. poly.

$$e_j(X_1, \dots, X_k) = \sum_{i_1 < \dots < i_j} X_{i_1} \cdots X_{i_j}$$

Then $\gamma = c_k, \alpha_i = y_i$ will do.

Step 2 Find a comm. F -alg A s.t. any nonconst.

$f(X) \in F[X]$ can be decomposed as

$$f(X) = \gamma (X - \alpha_1) \cdots (X - \alpha_k), \quad \alpha_i, \gamma \in A$$

How: take the tensor prod. $\bigotimes_{\substack{f(X) \in F[X] \\ \text{nonconst}}} A_f$

Step 3 Find a field E , homomorphism $A \rightarrow E$

How: take a maximal ideal $M \subset A$

$$\text{Set } E = A/M$$