

§ 49 Extension of field isomorphism

Suppose we have field extensions $F \subset E, E'$

When can we compare them, by homomorphism

$E \xrightarrow{\phi} E'$? (we want $\phi(a) = a$ for $a \in F$)

- E' should be big enough so that each $\alpha \in E$ has a candidate for $\phi(\alpha)$: if $f(\alpha) = 0$ for some $f(x) \in F[x]$, then $\exists \beta \in E'$ s.t. $f(\beta) = 0$

Thm 49.3 If $E \supset F$ is an algebraic ext., E' is algebraically closed, then the above question has a positive answer

Analogy from geometry

- finite covers of circle $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$
 $\phi_k: S^1 \rightarrow S^1, z \mapsto z^k \quad (\Leftrightarrow F \subset E)$
 universal cover $\psi: \mathbb{R} \rightarrow S^1, t \mapsto e^{2\pi i t} \quad (\Leftrightarrow F \subset E')$
 $\psi_k: \mathbb{R} \rightarrow S^1, t \mapsto e^{\frac{2\pi i t}{k}}$ satisfies $\phi_k \circ \psi_k = \psi$
 $(\Leftrightarrow E \rightarrow E')$
- uniformization of Riemann surfaces

Proof of Thm 49.3

Idea: take $\alpha_1 \in E$ (not in F) then

$\alpha_2 \in E$ not in $F(\alpha_1) = E_1$

$\alpha_3 \in E$ not in $F(\alpha_1, \alpha_2) = E_2, \dots$

Find $\phi_k: E_k = F(\alpha_1, \dots, \alpha_k) \rightarrow E'$ by induction

ϕ_1 : find $\beta_1 \in E'$ s.t. $f_1(\beta_1) = 0$ for the min. poly. $f_1(x) \in F[x]$ for $\alpha_1 \rightsquigarrow \phi_1(\alpha_1) = \beta_1$

ϕ_2 : take $f_2(x) \in E_1[x]$ min. poly. of α_2 over E_1 Then find $\beta_2 \in E'$ s.t. $\phi_1(f_2)(\beta_2) = 0$

→ continue this to get $\phi_k \upharpoonright_{E_{k-1}} = \phi_{k-1}$
and set $\phi(\beta) = \phi_k(\beta)$
if $\beta \in E_k$

Formal proof: use Zorn's lemma to

$\mathcal{C} = \{ (L, \lambda) \mid E \supset L \supset F \text{ subextension, } \lambda: L \rightarrow E' \text{ hom. } \lambda(a) = a \ (a \in F) \}$
with order $(L, \lambda) \leq (L', \lambda')$ if $L \subset L'$ & λ' extends λ

Rem. more formally $F \xrightarrow{\psi} E, F \xrightarrow{\psi'} E'$ hom. of fields

- E alg. ext. of $\psi(F)$
- E' algebraically closed ~~over $\psi(F)$~~

→ we can find $\phi: E \rightarrow E'$ s.t. $\phi(\psi(a)) = \psi'(a)$
 $a \in F$

Conseq. if $\alpha, \beta \in E$ are conj. over F

(i.e. same ~~irred~~ min. polynom) ^{alg. clos.} (and $\bar{F} \supset E \supset F$ or $F' \supset E \supset F$)
then $\exists \phi: E \rightarrow \bar{F}$ s.t. $\phi|_{F(\alpha)} = \psi_{\alpha, \beta}$
(or $E \rightarrow F'$) which sends α to β

Ex. $E = \mathbb{Q}(\sqrt{2} + \sqrt{3}) = \mathbb{Q}(\sqrt{2}, \sqrt{3})$

$\alpha = \sqrt{2}, \beta = -\sqrt{2}$ are conj. over $\mathbb{Q}$

$\sigma(\sqrt{2} + \sqrt{3}) = -\sqrt{2} + \sqrt{3}$ extends to $E \xrightarrow{\sigma} E$

($\sqrt{2} + \sqrt{3}$ and $-\sqrt{2} + \sqrt{3}$ are conj. over $\mathbb{Q}$)

σ as $E \rightarrow \mathbb{C}$ extends $\psi_{\alpha, \beta}$

Uniqueness of algebraic closures

$\bar{F} \supset F$ is an algebraic closure if

- $\bar{F} \supset F$ is an alg. ext.
- $\bar{F}$ is algebraically closed

If F' is another such extension

we have $\phi: \bar{F} \xrightarrow{\sim} F'$ isom. of extensions

We will fix $\bar{F} \supset F$

Def. $E \supset F$ finite extension

the number of homs $E \xrightarrow{\phi} \bar{F}$ s.t. $\phi(a) = a$
($a \in F$)
is called the index of E over F

write $\{E:F\}$

In fact : if F contains $\mathbb{Q}$ or $\mathbb{F}_p$ (is a) finite field

then $\{E:F\} = [E:F] = \dim_F E$

Ex. $E = \mathbb{Q}(\sqrt{2} + \sqrt{3})$ has 4 homs to $\bar{\mathbb{Q}}$ (or $\mathbb{C}$)

σ_0 = inclusion map

$$\sigma_1(\sqrt{2} + \sqrt{3}) = -\sqrt{2} + \sqrt{3}, \quad \sigma_2(\sqrt{2} + \sqrt{3}) = \sqrt{2} - \sqrt{3}$$

$$\sigma_3(\sqrt{2} + \sqrt{3}) = -\sqrt{2} - \sqrt{3}$$

1. The first part of the paper discusses the importance of maintaining accurate records of all transactions. It emphasizes that proper record-keeping is essential for the success of any business or organization. The author provides a detailed explanation of the various methods used to collect and analyze data, highlighting the importance of consistency and accuracy in the process.

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