

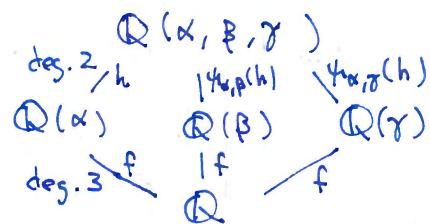
§50 Splitting fields

Motivating example : fields from $x^3 = 2$ candidates for x (roots of $f(x) = x^3 - 2$) :the real number $\sqrt[3]{2} = 1.2599\dots = \alpha$ imaginary numbers $e^{\frac{2\pi i}{3}} \sqrt[3]{2}$, $e^{-\frac{2\pi i}{3}} \sqrt[3]{2} = \beta, \gamma$ we have isoms. $\mathbb{Q}(\alpha) \simeq \mathbb{Q}(\beta) \simeq \mathbb{Q}(\gamma)$

$$g(\alpha) \leftrightarrow g(\beta) \leftrightarrow g(\gamma) \text{ for } g(x) \in \mathbb{Q}[x]$$

but - as subsets of $\mathbb{C}$, they are different- bigger field $\mathbb{Q}(\alpha, \beta, \gamma) (= \mathbb{Q}(\alpha, \beta))$

is another sensible system

related issue $x^3 - 2 = (x - \sqrt[3]{2}) \underbrace{(x^2 + \sqrt[3]{2}x + \sqrt[3]{2}^2)}_{h(x) : \text{irred. in } \mathbb{Q}(\alpha)}$ Why should we consider $\mathbb{Q}(\alpha, \beta, \gamma)$?any homomorphism $\varphi : \mathbb{Q}(\alpha, \beta, \gamma) \rightarrow \mathbb{C}$ has theimage $\mathbb{Q}(\alpha, \beta, \gamma)$ by $\varphi(\alpha) \in \{\alpha, \beta, \gamma\}$, hence
root of f φ comes from an autom. of $\mathbb{Q}(\alpha, \beta, \gamma)$
(symmetry)Def. $F' \supset F$ algebraically closed extension
(50.1) $S = \{f_i(x) \in F[x] \mid i \in I\}$ some collection of polynomialssubfield $E \subset F'$ (containing F) is thesplitting field of S over F if $E = F(\alpha \mid \alpha \text{ is a root of } f_i(x) \text{ for some } i \in I)$ (we are taking all possible roots of all $f_i(x)$)

Rem - E is an algebraic ext. ($\subset$ alg. clos. of F)

- up to isom, does not dep. on the choice of F'

Ex. $\mathbb{Q}(\sqrt[3]{2})$ is not a splitting field over $\mathbb{Q}$

$\mathbb{Q}(\sqrt[3]{2}, e^{\frac{2\pi i}{3}} \sqrt[3]{2})$ is a sp. field of $\{X^3 - 2\}$

Thm $F \subset E \subset F'$ extensions, F' alg. closed.

Then E is a splitting field of some $S \subset F[X]$

if and only if any autom. $F' \xrightarrow{\sigma} F'$ s.t. $\sigma(a) = a$ for $a \in F$ satisfies $E = \sigma(E)$

Proof of " $\Rightarrow$ " direction

Assume $E = F(\alpha) \mid \alpha \text{ is a root of some } f(X) \in S$

and fix $\sigma : F' \rightarrow F'$ s.t. $\sigma(a) = a$ for $a \in F$

Step 1 If $f(X) \in F[X]$ α is a root of $f(X)$

$\Leftrightarrow \sigma(\alpha)$ is a root of $f(X)$

$$\sigma(c_0 + c_1 \alpha + \dots + c_n \alpha^n) = c_0 + c_1 \sigma(\alpha) + \dots + c_n \sigma(\alpha)^n$$

for $c_i \in F$

Step 2 $\sigma(E) = E$

$\sigma(E) = F(\sigma(\alpha) \mid \alpha \text{ is a root of } \dots)$

Proof of " $\Leftarrow$ " direction

Step 1 If $\alpha \in E$, $f(X)$ min. polynom. of α over F

and β is another root of $f(X)$, then

$\beta \in E$

$\psi_{\alpha, \beta} : F(\alpha) \rightarrow F(\beta)$ would extend to $F' \xrightarrow{\sigma} F'$

and $E = \sigma(E) \leadsto \beta = \psi_{\alpha, \beta}(\alpha) = \sigma(\alpha)$ is in E

Step 2 E is a splitting field of

$S = \{f(X) \in F[X] \mid \text{min. polynom. of some } \alpha \in E\}$

Cor. When $E \supset F$ is a splitting extension,

$$G(E/F) = \{ \varphi: E \rightarrow E \mid \text{aut.}, \varphi(a) = a \text{ for } a \in F \}$$

has the same num. of elements as

$$\{ E \xrightarrow{\psi} \bar{F} \mid \text{hom}, \psi(a) = a \text{ for } a \in F \}$$

$$\text{i.e. } |G(E/F)| = \{ E : F \}$$

Fact (next lecture) If $F \supset \mathbb{Q}$ then

$$\{ E : F \} = [E : F]$$

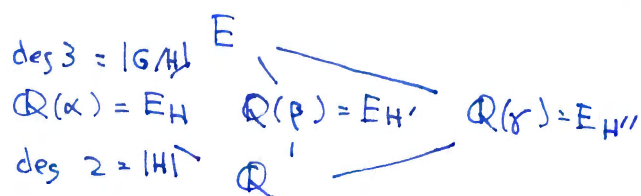
so $E = \mathbb{Q}(\sqrt[3]{2}, e^{\frac{2\pi i}{3}} \sqrt[3]{2}) \supset \mathbb{Q}$ has deg. 6

hence $G(E/\mathbb{Q})$ is order 6

We have $G(E/\mathbb{Q}) \cong S_3$ permuting
 $\alpha = \sqrt[3]{2}, \beta = e^{\frac{2\pi i}{3}} \sqrt[3]{2}, \gamma = e^{-\frac{2\pi i}{3}} \sqrt[3]{2}$

subgroup $H = \{ \text{id}, \sigma \mid \sigma(\alpha) = \bar{\alpha} \text{ (complex conj.)} \}$
 $\sigma(\beta) = \gamma, \sigma(\gamma) = \beta.$

The fixed-point set $\{ b \in E \mid \sigma(b) = b \} = E_H$
 is $\mathbb{Q}(\alpha)$. So we have





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1. The first part of the paper is devoted to a general discussion of the problem.

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2. In the second part, we consider the case of a single particle.

3. The third part is devoted to the case of a system of particles.

3

4. In the fourth part, we consider the case of a continuous medium.

5. The fifth part is devoted to the case of a system of continuous media.

6. In the sixth part, we consider the case of a system of particles and continuous media.

4

7. The seventh part is devoted to the case of a system of particles and continuous media.

8. In the eighth part, we consider the case of a system of particles and continuous media.

9. The ninth part is devoted to the case of a system of particles and continuous media.

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10. In the tenth part, we consider the case of a system of particles and continuous media.

11. The eleventh part is devoted to the case of a system of particles and continuous media.

12. In the twelfth part, we consider the case of a system of particles and continuous media.

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13. The thirteenth part is devoted to the case of a system of particles and continuous media.

14. In the fourteenth part, we consider the case of a system of particles and continuous media.

15. The fifteenth part is devoted to the case of a system of particles and continuous media.

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16. In the sixteenth part, we consider the case of a system of particles and continuous media.

17. The seventeenth part is devoted to the case of a system of particles and continuous media.

18. In the eighteenth part, we consider the case of a system of particles and continuous media.

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19. The nineteenth part is devoted to the case of a system of particles and continuous media.

20. In the twentieth part, we consider the case of a system of particles and continuous media.

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21. The twenty-first part is devoted to the case of a system of particles and continuous media.

22. In the twenty-second part, we consider the case of a system of particles and continuous media.

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23. The twenty-third part is devoted to the case of a system of particles and continuous media.

24. In the twenty-fourth part, we consider the case of a system of particles and continuous media.

11

25. The twenty-fifth part is devoted to the case of a system of particles and continuous media.

26. In the twenty-sixth part, we consider the case of a system of particles and continuous media.

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