

MOCK EXAM PROBLEMS FOR MAT2200

All problems have the same weight. Note: You must give full reasons for all your answers.

Problem 1. For each pair of groups among the following list, determine if they are isomorphic or not. If they are isomorphic, give an isomorphism. If they are not, give a reason why.

- $\{z \in \mathbb{C} \mid z^4 = 1\}$, with multiplication
- $\mathbb{Z}_4$
- $\mathbb{Z}_5$

Answer. The group $G = \{z \in \mathbb{C} \mid z^4 = 1\}$ and $\mathbb{Z}_4$ are isomorphic. Concretely, we define a map from $\mathbb{Z}_4$ to G by $\phi([k]) = e^{\frac{2\pi i k}{4}}$. This is well-defined, since $[k] = [\ell]$ in $\mathbb{Z}_4$ means $k - \ell = 4m$ for some integer m , and this implies

$$e^{\frac{2\pi i k}{4}} = e^{\frac{2\pi i (\ell + 4m)}{4}} = e^{\frac{2\pi i \ell}{4}} e^{2\pi i m} = e^{\frac{2\pi i \ell}{4}}.$$

We have

$$\phi([k])\phi([\ell]) = e^{\frac{2\pi i k}{4}} e^{\frac{2\pi i \ell}{4}} = e^{\frac{2\pi i (k+\ell)}{4}} = \phi([k+\ell]) = \phi([k] + [\ell]),$$

which shows that ϕ is a homomorphism of groups. We know that $\mathbb{Z}_4$ and G both have 4 elements, and ϕ is an onto map. Then ϕ must be also one-to-one, hence it is an isomorphism.

The groups $\mathbb{Z}_4$ and $\mathbb{Z}_5$ are not isomorphic, since they have different number of elements, and in particular there cannot be a onto map from $\mathbb{Z}_4$ to $\mathbb{Z}_5$.

Since we know that G and $\mathbb{Z}_4$ are isomorphic, while $\mathbb{Z}_4$ and $\mathbb{Z}_5$ are not, G and $\mathbb{Z}_5$ are not isomorphic. □

Problem 2. Consider the following maps from $\mathbb{Z}^2$ to $\mathbb{Z}$. Determine if they are group homomorphisms or not.

- addition map $f(x, y) = x + y$
- multiplication map $g(x, y) = xy$

Answer. The addition map $f(x, y)$ is a homomorphism. We have

$$f(x, y) + f(x', y') = (x + y) + (x' + y') = x + x' + y + y' = f(x + x', y + y'),$$

where we used the associativity and the commutativity of addition in the second equality.

The multiplication map $g(x, y)$ is not a homomorphism. If it is a homomorphism, we should have

$$xy + x'y' = g(x, y) + g(x', y') = g(x + x', y + y') = (x + x')(y + y'),$$

but the choices $(x, y) = (1, 0)$ and $(x', y') = (0, 1)$ violates this equality. □

Problem 3. Consider the subgroup $H \subset S_5$, given as

$$H = \{\sigma \in S_5 \mid \sigma(i) \in \{1, 2\} \text{ for } i \in \{1, 2\}, \sigma(i) \in \{3, 4, 5\} \text{ for } i \in \{3, 4, 5\}\}.$$

Determine if H is normal or not.

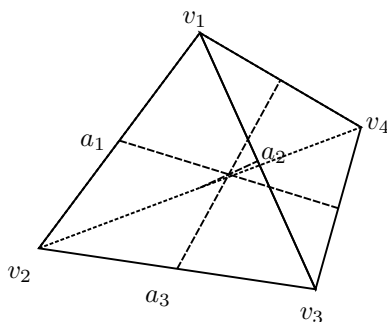
Answer. The subgroup H is not normal. Consider the element $\sigma = (1, 2)(3, 4) \in H$, and the element $\tau = (2, 3) \in S_5$. Then the element $\tau\sigma\tau^{-1}$ sends 2 to

$$\tau(\sigma(\tau^{-1}(2))) = \tau(\sigma(3)) = \tau(4) = 4,$$

hence it is not an element of H . □

Problem 4. Consider the action of S_4 on tetrahedron, where each $\sigma \in S_4$ sends the vertex v_i to $v_{\sigma(i)}$ for $i = 1, 2, 3, 4$. Consider the subgroup of permutations that induce rotations around one of the axis a_1, a_2, a_3 . (We allow the trivial permutation too.) What is the order of this group?

Answer. The 180° rotation around a_1 flips v_1 and v_2 , and simultaneously v_3 and v_4 , hence it is represented by the element $(1, 2)(3, 4) \in S_4$. Similarly, the 180° rotation around a_2 is represented by $(1, 3)(2, 4)$, and the 180° rotation around a_3 is represented by $(1, 4)(2, 3)$. Together with the trivial permutation, the subgroup in question is $H = \{e, (1, 2)(3, 4), (1, 3)(2, 4), (1, 4)(2, 3)\}$, isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. Hence it has 4 elements. □



Problem 5. Determine if the following rings are integral domains or not.

- $\mathbb{Z}_5$
- $\mathbb{Z}_6$

Answer. The ring $\mathbb{Z}_5$ is an integral domain, since $[a][b] = 0$ in $\mathbb{Z}_5$ means $ab = 5k$ for some integer k , which forces 5 to be a prime factor of a or b , hence $[a] = 0$ or $[b] = 0$.

The ring $\mathbb{Z}_6$ is not an integral domain. Setting $a = 2$ and $b = 3$, we have $[a] \neq 0$ and $[b] \neq 0$ since 6 does not divide 2 nor 3. However, we have $[a][b] = [6] = 0$. $\square$

Problem 6. What are the maximal ideals of $\mathbb{Z}_6$?

Answer. The ideals of $\mathbb{Z}_6$ correspond to the divisors of 6, hence

$$0\mathbb{Z}_6 = \{[0]\}, \quad 2\mathbb{Z}_6 = \{[0], [2], [4]\}, \quad 3\mathbb{Z}_6 = \{[0], [3]\}, \quad 1\mathbb{Z}_6 = \{[0], [1], \dots, [5]\} = \mathbb{Z}_6.$$

The maximal ideals are $2\mathbb{Z}_6$ and $3\mathbb{Z}_6$. $\square$

Problem 7. Consider the polynomial $f(X) = X^2 + 1$ in $\mathbb{Q}[X]$. Determine if

$$\{f(X)g(X) \mid g(X) \in \mathbb{Q}[X]\} \subset \mathbb{Q}[X]$$

is a prime ideal or not.

Answer. The complex roots of $f(X)$ are i and $-i$, which are not in $\mathbb{Q}$. This means that $f(X)$ does not factorize as a product of lower order polynomials (which should be of degree 1) in $\mathbb{Q}[X]$, hence is irreducible. Then the corresponding ideal as in the problem is a prime ideal. $\square$

Problem 8. Consider the polynomial $f(X) = X^2 + 1$ in $\mathbb{Z}_5[X]$. Determine if

$$\{f(X)g(X) \mid g(X) \in \mathbb{Z}_5[X]\} \subset \mathbb{Z}_5[X]$$

is a prime ideal or not.

Answer. There are roots of $f(X)$ in $\mathbb{Z}_5$, namely, 2 and 3, since we have $2^2 = 4 = -1$ in $\mathbb{Z}_5$, and similarly for $3^2 = 9$. We thus have a factorization $f(X) = (X - 2)(X - 3)$ in $\mathbb{Z}_5[X]$, and in particular $g(X) = X - 2$ and $h(X) = X - 3$ satisfy $g(X), h(X) \notin I_f$ while $g(X)h(X) = f(X) \in I_f$, where I_f is the ideal in question. This shows I_f is not prime. $\square$

Problem 9. What is the degree of extension of $\mathbb{Q}(\sqrt{5})$ over $\mathbb{Q}$?

Answer. The minimal polynomial for $\sqrt{5}$ over $\mathbb{Q}$ is $f(X) = X^2 - 5$. We thus have $[\mathbb{Q}(\sqrt{5}) : \mathbb{Q}] = \deg f(X) = 2$. Concretely, 1 and $\sqrt{5}$ form a basis of $\mathbb{Q}(\sqrt{5})$ as a vector space over $\mathbb{Q}$. $\square$

Problem 10. Consider the extension $\mathbb{Q}(\sqrt[3]{3}) \supset \mathbb{Q}$. Show that this is not a splitting extension.

Answer. The minimal polynomial for $\sqrt[3]{3}$ is $f(X) = X^3 - 3$, and its roots are $\alpha = \sqrt[3]{3}$, $\beta = e^{\frac{2\pi i}{3}} \sqrt[3]{3}$, and $\gamma = e^{-\frac{2\pi i}{3}} \sqrt[3]{3}$. Since $\mathbb{Q}(\sqrt[3]{3})$ is a subfield of $\mathbb{R}$, it does not contain β and γ , and is not a splitting extension of $\mathbb{Q}$. $\square$

Problem 11. Let $f(X)$ be the minimal polynomial of $\sqrt[3]{3}$ over $\mathbb{Q}$. Determine the splitting extension $E \supset \mathbb{Q}$ for $f(X)$.

Answer. With the same notation as above, $E = \mathbb{Q}(\alpha, \beta, \gamma)$ is the splitting field of $f(X)$ over $\mathbb{Q}$. This contains $F = \mathbb{Q}(\alpha) = \mathbb{Q}(\sqrt[3]{3})$ as a subextension, and we have $E = F(\beta)$ since γ can be obtained from α and β by $\gamma = \beta^2/\alpha$. The minimal polynomial of β over F is $f(X)/(X - \alpha)$, which is of degree 2. This shows that E is of degree 2 over F , hence E is a field generated by α and β over $\mathbb{Q}$, and of degree 6. $\square$

Problem 12. Determine the automorphism group of the extension $G(E/\mathbb{Q})$ for the extension $E \supset \mathbb{Q}$ above.

Answer. Solution 1:

Elements σ in $G(E/\mathbb{Q})$ induce permutations of the set of roots of $f(X)$, namely $S = \{\alpha, \beta, \gamma\}$. Since E is generated by α, β, γ , over $\mathbb{Q}$, the effect of σ on E is completely determined by its effect on S . Since E is a splitting field of characteristic 0, it is a Galois extension and we have $|G(E/\mathbb{Q})| = [E : \mathbb{Q}] = 6$. This means that $G(E/\mathbb{Q})$ can be realized as a subgroup of order 6 in S_3 . Since the order of S_3 is also 6, we have $G(E/\mathbb{Q}) \simeq S_3$.

Solution 2:

The order of $G(E/\mathbb{Q})$ is $6 = [E : \mathbb{Q}]$, since E is a splitting field of characteristic 0, hence is a Galois extension. Let us check that $G(E/\mathbb{Q})$ is noncommutative. On one hand, it has the complex conjugation map τ , which flips β and γ and fixes α . On the other hand, there is an element $\sigma \in G(E/\mathbb{Q})$ which is an extension of $\psi_{\alpha, \beta}: \mathbb{Q}(\alpha) \rightarrow \mathbb{Q}(\beta)$ that sends α to β . Then $\sigma\tau$ sends α to β while $\tau\sigma$ sends α to γ . Hence $G(E/\mathbb{Q})$ is a noncommutative group of order 6, hence it must be isomorphic to S_3 . $\square$